

Fluid Simulation for Computer Animation

Why Simulate Fluids?

- Feature film special effects
- Computer games
- Medicine (e.g. blood flow in heart)
- Because it's fun

Fluid Simulation

- Called *Computational Fluid Dynamics* (CFD)
- Many approaches from math and engineering
- Graphics favors *finite differences*
- Jos Stam introduced fast and stable methods to graphics [Stam 1999]

Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0 \quad \text{Incompressibility}$$

$$\mathbf{u}_t = \underbrace{\mathbf{k} \nabla^2 \mathbf{u}}_{\text{Diffusion}} - \underbrace{(\mathbf{u} \cdot \nabla) \mathbf{u}}_{\text{Advection}} - \underbrace{\nabla p}_{\text{Pressure}} + \underbrace{\mathbf{f}}_{\text{Body Forces}}$$

The diagram shows the Navier-Stokes equation with five terms. Each term is enclosed in a teal bracket. A teal vertical line with a dot at the bottom connects each bracket to its corresponding label: 'Change in Velocity' for \mathbf{u}_t , 'Diffusion' for $\mathbf{k} \nabla^2 \mathbf{u}$, 'Advection' for $(\mathbf{u} \cdot \nabla) \mathbf{u}$, 'Pressure' for ∇p , and 'Body Forces' for \mathbf{f} .

Change in Velocity

Diffusion

Advection

Pressure

Body Forces

Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0 \quad \text{Incompressibility}$$

$$\mathbf{u}_t = \underbrace{\mathbf{k} \nabla^2 \mathbf{u}}_{\text{Diffusion}} - \underbrace{(\mathbf{u} \cdot \nabla) \mathbf{u}}_{\text{Advection}} - \underbrace{\nabla p}_{\text{Pressure}} + \underbrace{\mathbf{f}}_{\text{Body Forces}}$$

Change in Velocity

Body Forces

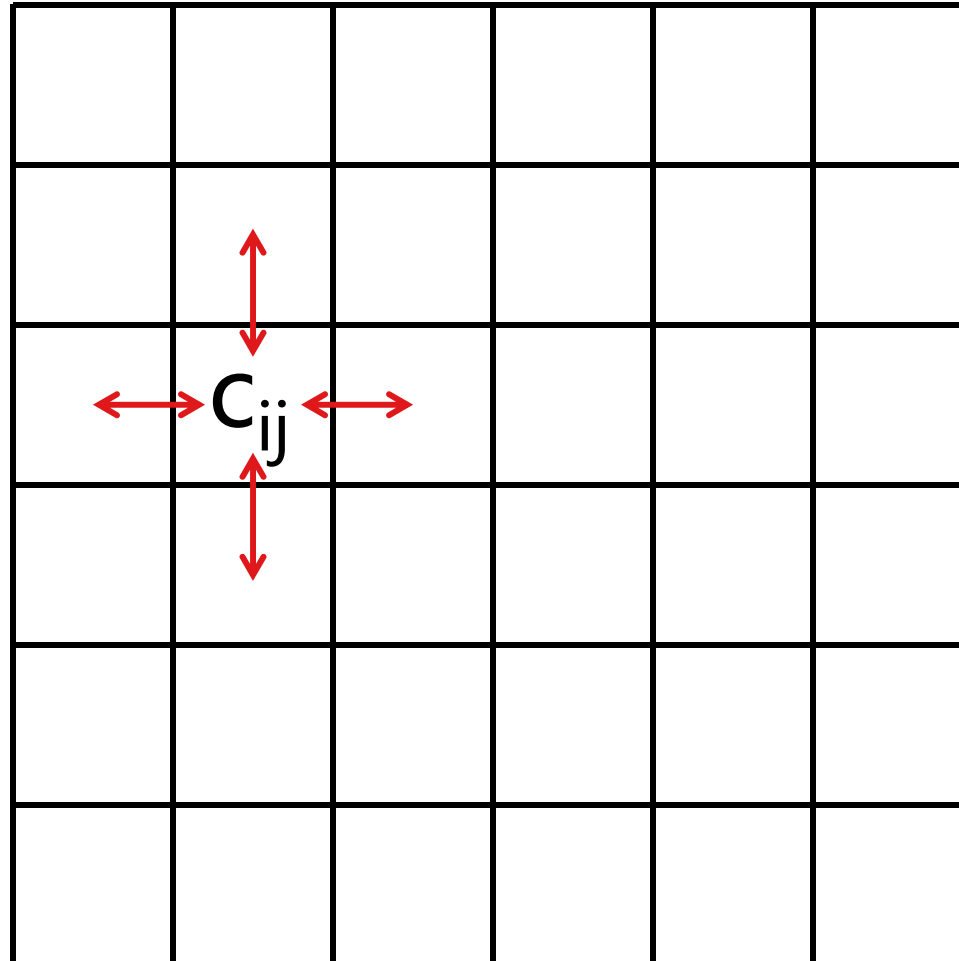
Finite Differences Grids

- All values live on regular grids
- Need *scalar* and *vector* fields
- Scalar fields: amount of smoke or dye
- Vector fields: fluid velocity
- Subtract adjacent quantities to approximate derivatives

Scalar Field (Smoke, Dye)

1.2	3.7	5.1	...		
	c_{ij}				

Diffusion

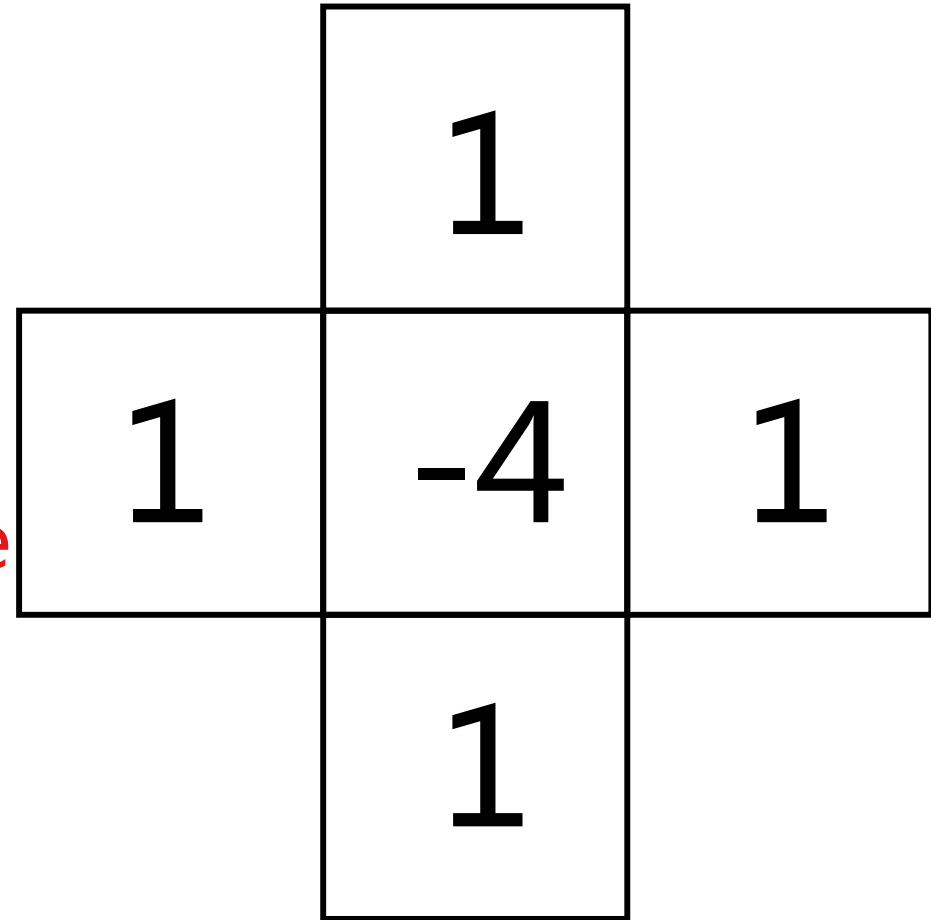


Diffusion

$$c_t = k \nabla^2 c$$

change in value

value relative to neighbors



$$c_{ij}^{\text{new}} = c_{ij} + k \Delta t (c_{i-1j} + c_{i+1j} + c_{ij-1} + c_{ij+1})$$

Diffusion = Blurring



Original

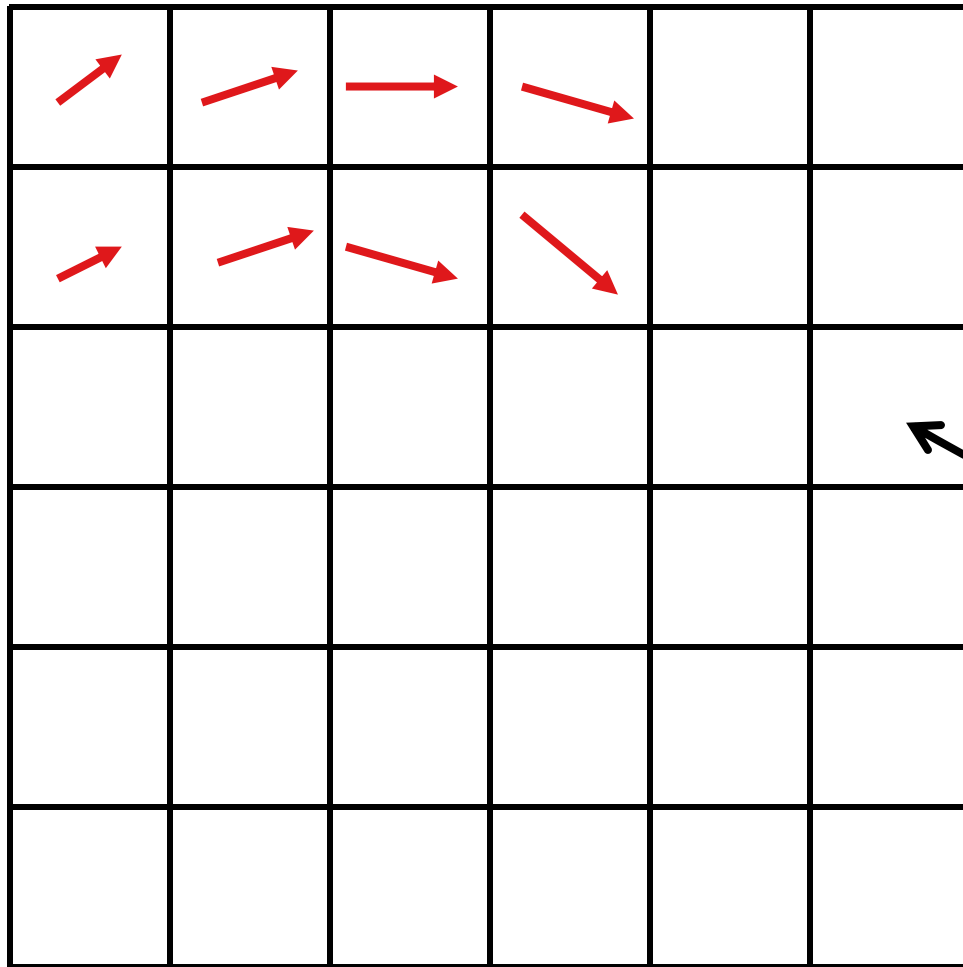


Some Diffusion



More Diffusion

Vector Fields (Fluid Velocity)




$$\mathbf{u}_{ij} = (u^x, u^y)$$



Vector Field Diffusion

$$\mathbf{u}_t = k \nabla^2 \mathbf{u}$$

 viscosity

Two separate diffusions:

$$u^x_t = k \nabla^2 u^x$$

$$u^y_t = k \nabla^2 u^y$$

... blur the x -velocity and the y -velocity

Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0 \quad \text{Incompressibility}$$

$$\mathbf{u}_t = \underbrace{\mathbf{k} \nabla^2 \mathbf{u}}_{\text{Diffusion}} - \underbrace{(\mathbf{u} \cdot \nabla) \mathbf{u}}_{\text{Advection}} - \underbrace{\nabla p}_{\text{Pressure}} + \underbrace{\mathbf{f}}_{\text{Body Forces}}$$

Change in Velocity

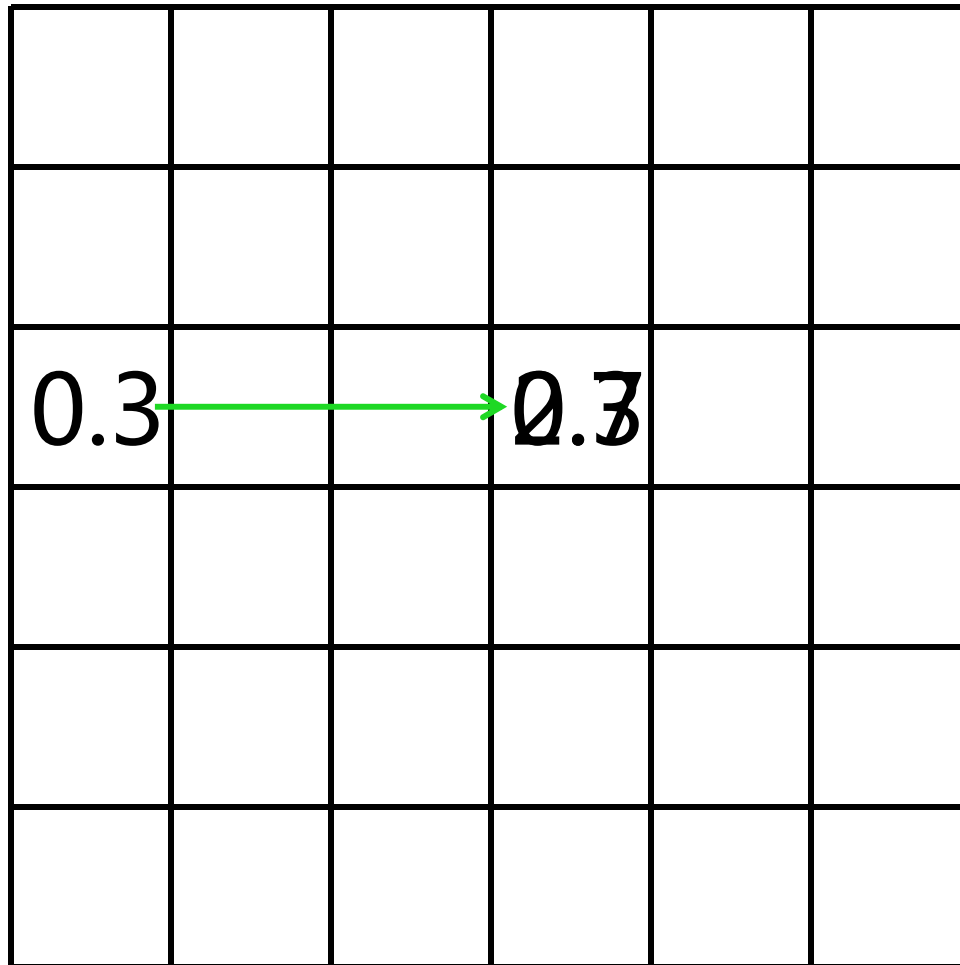
Diffusion

Advection

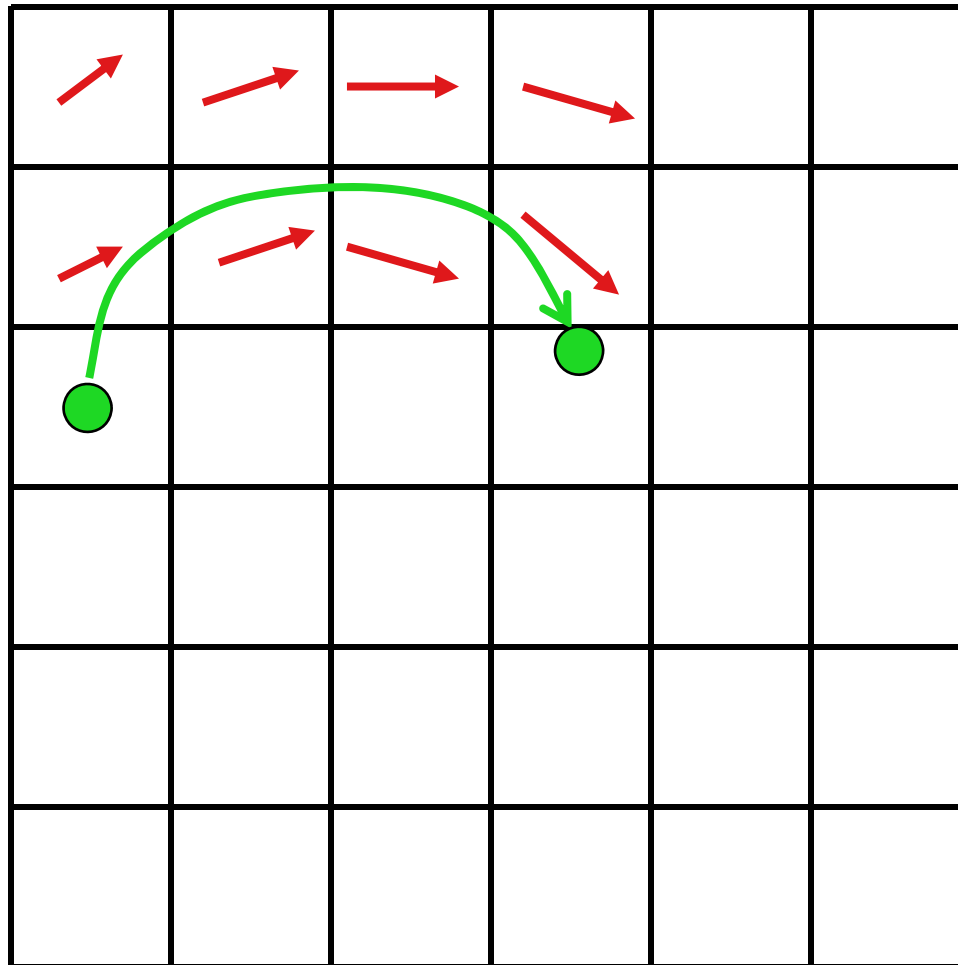
Pressure

Body Forces

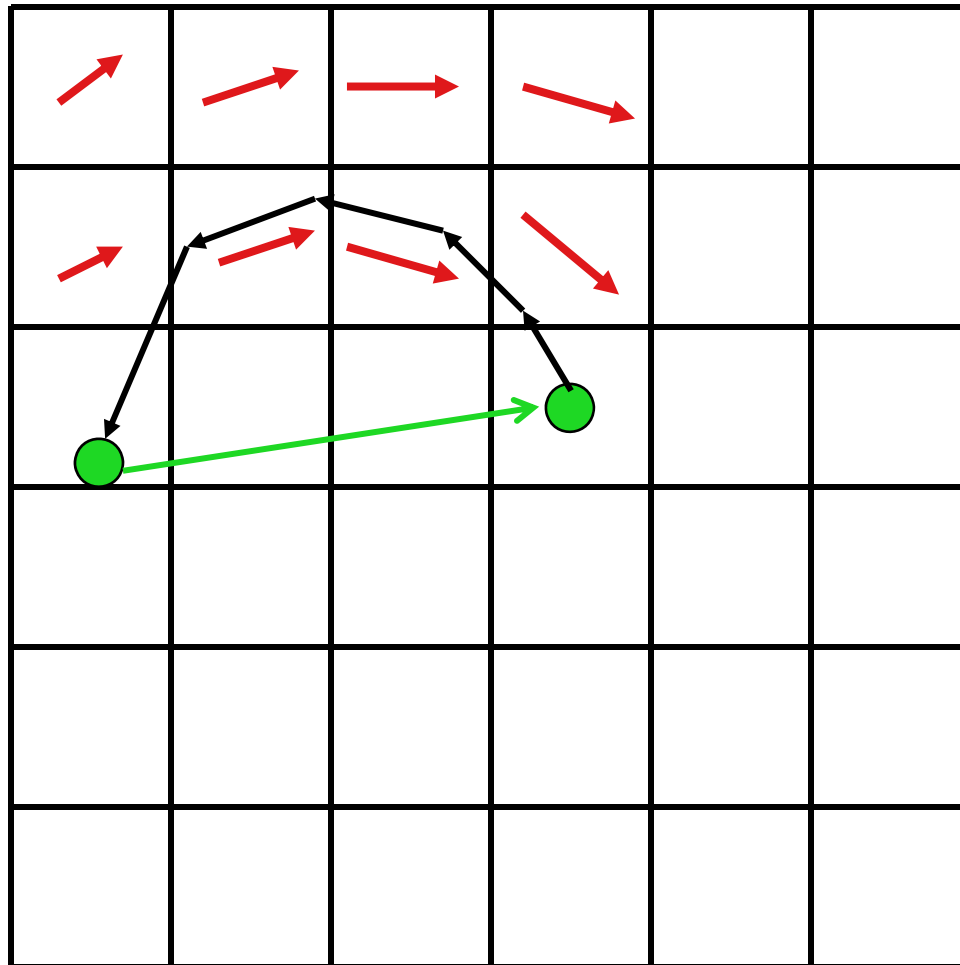
Advection: stuffs get pushed by the flow



Forward Advection



Backward Advection



Scalar Field Advection

$$c_t = -(\mathbf{u} \cdot \nabla) c$$

change in value

advection

current values

The diagram illustrates the components of the scalar field advection equation. Three red arrows point from descriptive text to parts of the equation: one from 'change in value' to the time derivative c_t , one from 'advection' to the divergence term $(\mathbf{u} \cdot \nabla)$, and one from 'current values' to the scalar field c .

Vector Field Advection

$$\mathbf{u}_t = -(\mathbf{u} \cdot \nabla) \mathbf{u}$$

Two separate advections:

$$u^x_t = -(\mathbf{u} \cdot \nabla) u^x$$

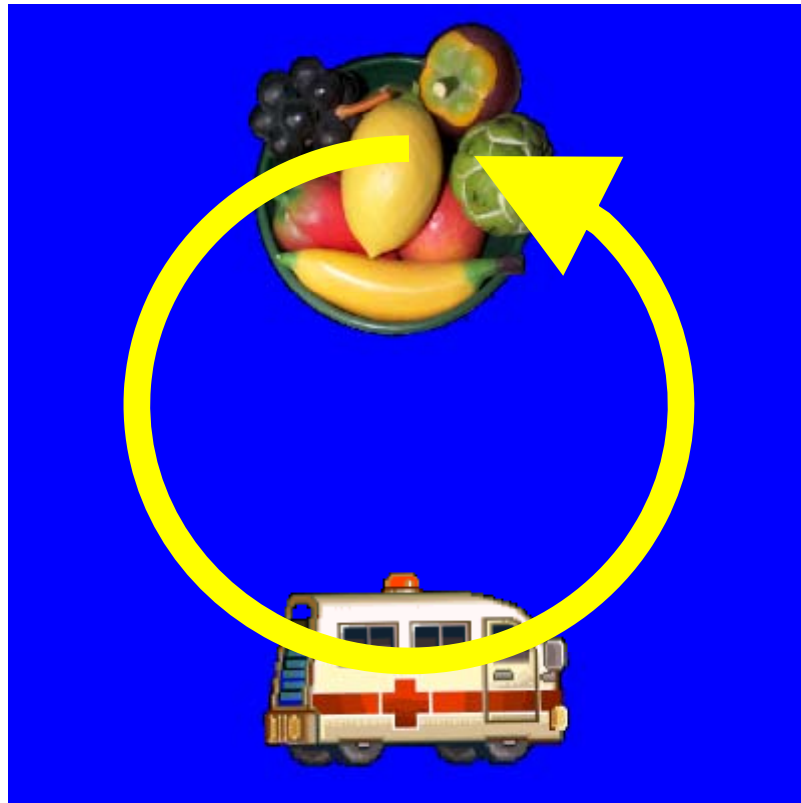
$$u^y_t = -(\mathbf{u} \cdot \nabla) u^y$$

... push around x -velocity and y -velocity

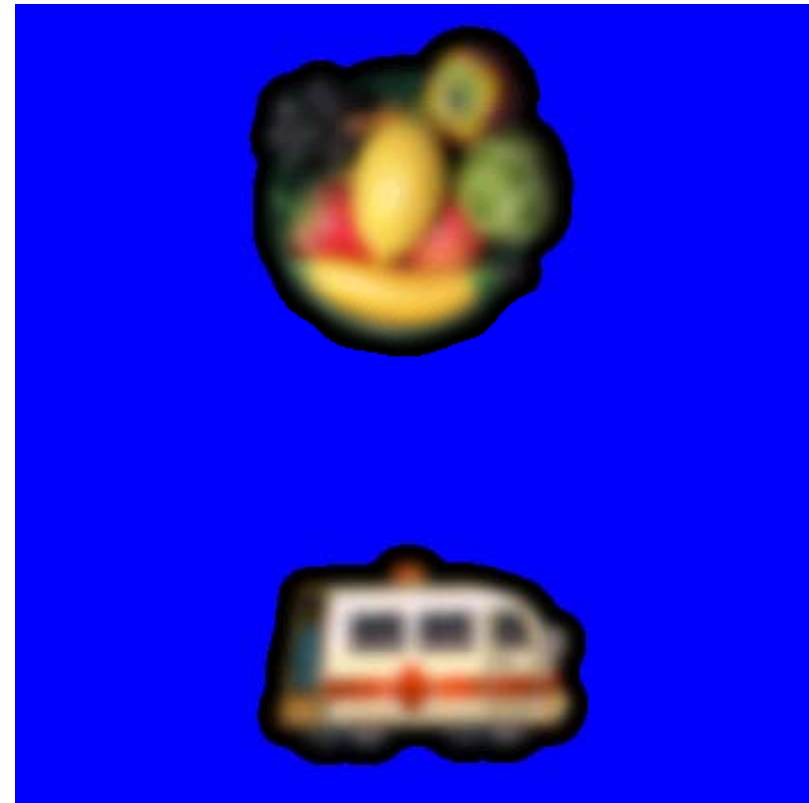
Advection

- Easy to code
- Method stable even at large time steps
- Problem: numerical inaccuracy diffuses flow

Diffusion/dissipation in first order advection



Original Image



After 360 degree rotation
using first order advection

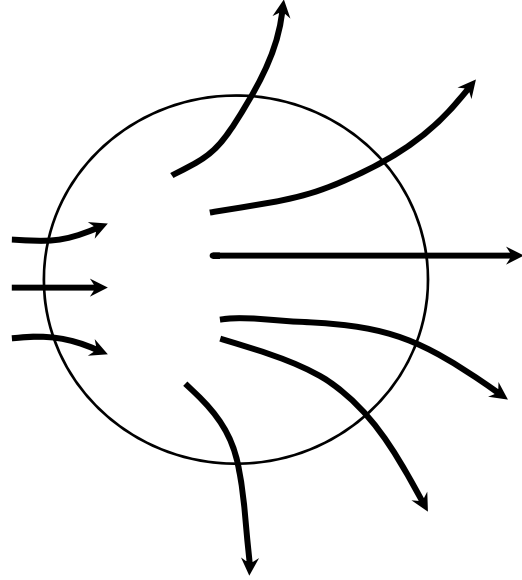
Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0 \quad \text{Incompressibility}$$

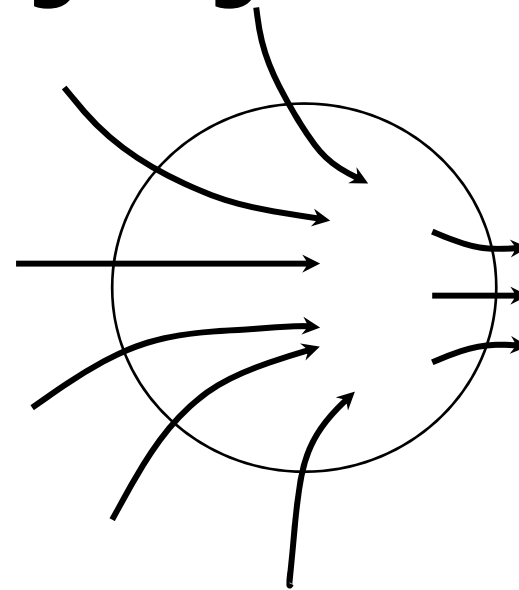
$$\mathbf{u}_t = \underbrace{\mathbf{k} \nabla^2 \mathbf{u}}_{\text{Diffusion}} - \underbrace{(\mathbf{u} \cdot \nabla) \mathbf{u}}_{\text{Advection}} - \underbrace{\nabla p}_{\text{Pressure}} + \underbrace{\mathbf{f}}_{\text{Body Forces}}$$

Change in Velocity

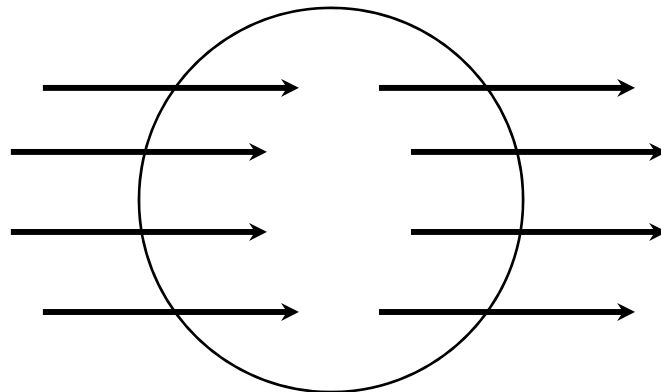
Divergence: Difference between incoming and outgoing flow



High divergence



Low divergence



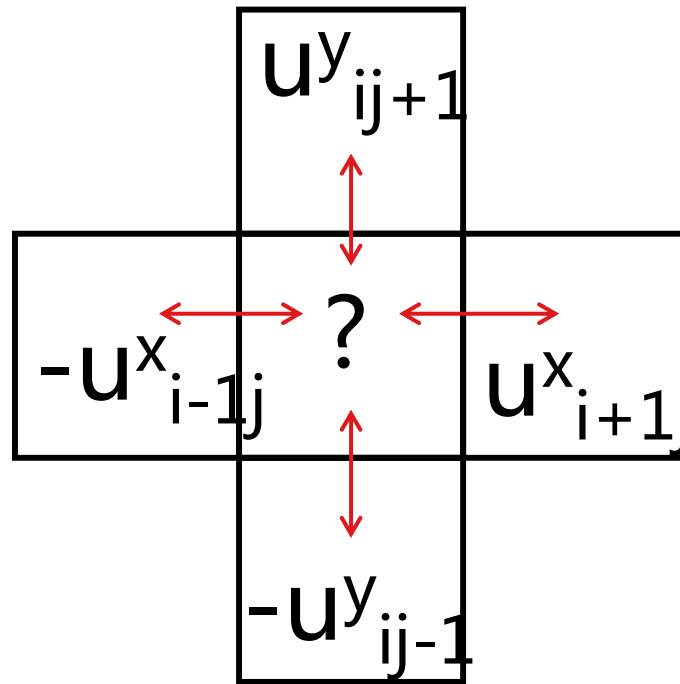
Zero divergence

Enforcing Incompressibility

- First do velocity diffusion and advection
- Find “closest” vector field that is divergence-free
- Need to calculate divergence
- Need to find and use pressure

Measuring Divergence

$$\nabla \cdot \mathbf{u} = ?$$



$$\nabla \cdot \mathbf{u}_{ij} = (u^x_{i+1j} - u^x_{i-1j}) + (u^y_{ij+1} - u^y_{ij-1})$$

Pressure Term

$$\mathbf{u}^{new} = \mathbf{u} - \nabla p$$

Take divergence of both sides...

$$\nabla \cdot \mathbf{u}^{new} = \nabla \cdot \mathbf{u} - \nabla \cdot \nabla p$$



zer

o

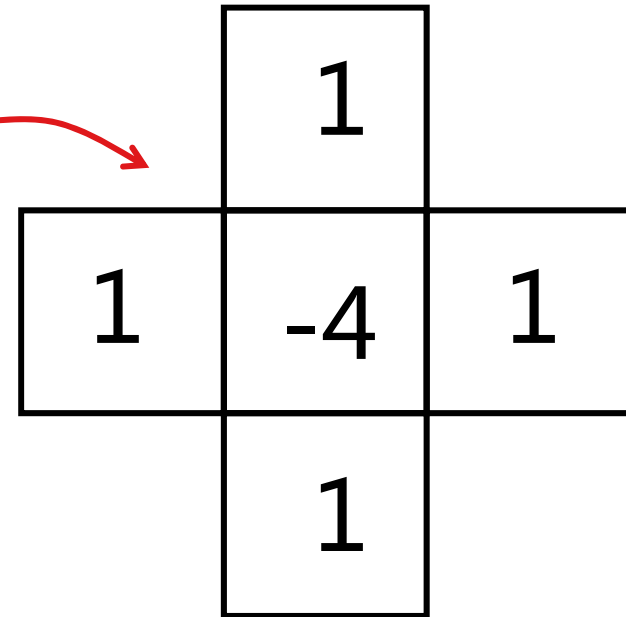
$$\nabla \cdot \mathbf{u} = \nabla^2 p$$

Pressure Term

$$\underbrace{\nabla \cdot \mathbf{u}}_{\text{known}} = \underbrace{\nabla^2 p}_{\text{unknown}}$$

known unknown

$$p^{\text{new}} = p + \varepsilon(\nabla \cdot \mathbf{u} - \nabla^2 p)$$



$$\text{Let } d_{ij} = \nabla \cdot \mathbf{u}_{ij}$$

$$p_{ij}^{\text{new}} = p_{ij} + \varepsilon(d_{ij} - (p_{i-1j} + p_{i+1j} + p_{ij-1} + p_{ij+1}) - 4p_{ij})$$

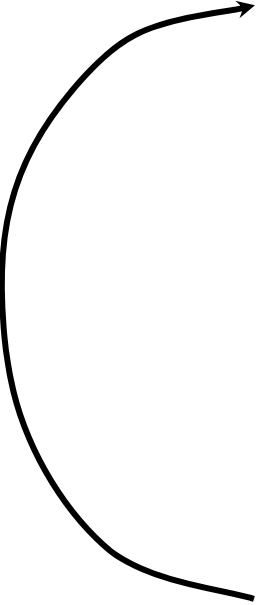
Pressure Term

$$\mathbf{u}^{new} = \mathbf{u} - \nabla p$$

...and velocity is now divergence-free

Found "nearest" divergence-free velocity field to original.

Fluid Simulator

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- 1) Diffuse velocity
 - 2) Advect velocity
 - 3) Add body forces (e.g. gravity)
 - 4) Pressure projection
 - 5) Diffuse dye/smoke
 - 6) Advect dye/smoke

“Fluid simulation”
SIGGRAPH 2006/2007 Course Note

<https://www.cs.ubc.ca/~rbridson/>

“Real-Time Fluid Dynamics for Games”
Jos Stam, March 2003
(CDROM link is to source code)

www.dgp.toronto.edu/people/stam/reality/Research/

Research Areas

- 1) Thin Features
- 2) Surface tension
- 3) Coupling
- 4) Fluid-like behaviors (sand, snow, et