The Jacobian

The Math

Most mechanisms of interest to computer animation are too complex to allow an analytic solution. For these, the motion can be incrementally constructed. At each time step, the best way to change each joint angle in order to direct the current position and orientation of the end effector toward the desired configuration is computed. The computation is preformed by forming the Jacobian matrix which is a matrix of partial derivatives.

In order to explain the Jacobian from a strictly mathematical point of view, consider the six arbitrary functions of EQ 1, each of which is a function of six independent variables. Given specific values for the input variables, the x_is, each of the output variables, the y_is, can be computed by its respective function.

$$\begin{aligned} y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\ y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6) \\ y_3 &= f_3(x_1, x_2, x_3, x_4, x_5, x_6) \\ y_4 &= f_4(x_1, x_2, x_3, x_4, x_5, x_6) \\ y_5 &= f_5(x_1, x_2, x_3, x_4, x_5, x_6) \\ y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6) \end{aligned} \quad \text{(EQ 1)}$$

The differentials of y_i can be written in terms of the differentials of x_i using the chain rule. The generates EQ 2.

$$\delta y_i = \frac{\delta f_i}{\partial x_1} \cdot \delta x_1 + \frac{\delta f_i}{\partial x_2} \cdot \delta x_2 + \frac{\delta f_i}{\partial x_3} \cdot \delta x_3 + \frac{\delta f_i}{\partial x_4} \cdot \delta x_4 + \frac{\delta f_i}{\partial x_5} \cdot \delta x_5 + \frac{\delta f_i}{\partial x_6} \cdot \delta x_6 \quad \text{(EQ 2)}$$

EQ 1 and EQ 2 can be put in vector notation producing EQ 3 and EQ 4.

$$Y = F(X) \quad \text{(EQ 3)}$$

$$\delta Y = \frac{\partial F}{\partial X} \cdot \partial X \quad \text{(EQ 4)}$$

The 6x6 matrix of partial derivatives, $\frac{\partial F}{\partial X}$, is called the *Jacobian* and is a function of the current values of the x_i. The Jacobian can be thought of mapping the velocities of X to the velocities of Y (EQ 5). At any particular point in time, the Jacobian is a linear function of the x_i's. At the next instant of time, X has changed and so has the linear transformation represented by the Jacobian.

$$\dot{Y} = J(X) \cdot \dot{X} \quad \textbf{(EQ 5)}$$

In applying the Jacobian to a linked appendage, the input variables, x_is, become the joint angles and the output variables, y_is, become the end effector position and orientation. In this case, the Jacobian relates the velocities of the joint angles to the velocities of the end effector position and orientation (EQ 6).

$$V = J(\theta)\dot{\theta} \quad \textbf{(EQ 6)}$$

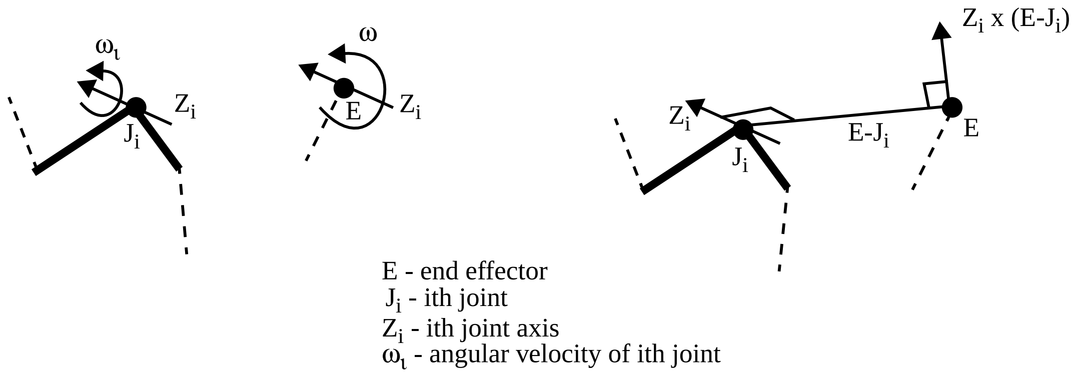
V is the vector of linear and rotational velocities and represents the desired change in the end effector. The desired change will be based on the difference between its current position/orientation to that specified by the goal configuration. These velocities are vectors in three-space so each has an x, y, and z component (EQ 7). $\dot{\theta}$ is a vector of joint angles velocities and are the unknowns of the equation (EQ 8). J, the Jacobian, is a matrix which relates the two and is a function of the current pose (EQ 9).

$$V = [v_x, v_y, v_z, \omega_x, \omega_y, \omega_z]^T \quad \textbf{(EQ 7)}$$

$$\dot{\theta} = [\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \ldots, \dot{\theta}_n]^T \quad \textbf{(EQ 8)}$$

$$J = \begin{bmatrix} \frac{\partial v_x}{\partial\theta 1} & \frac{\partial v_x}{\partial\theta 2} & \cdots & \frac{\partial v_x}{\partial\theta n} \\ \frac{\partial v_x}{\partial\theta 1} & \frac{\partial v_x}{\partial\theta 2} & \cdots & \frac{\partial v_x}{\partial\theta n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\omega_z}{\partial\theta 1} & \frac{\partial\omega_z}{\partial\theta 2} & \cdots & \frac{\partial\omega_z}{\partial\theta n} \end{bmatrix} \quad \textbf{(EQ 9)}$$

Each term of the Jacobian relates the change of a specific joint with a change in the end effector. The rotational change in the end effector, ω, is merely the velocity of the joint angle about the axis of revolution at the joint under consideration. The linear change in the end effector is the cross product of the axis of revolution and a vector from the joint to the end effector. This is the linear direction of travel instantaneously induced at the end effector by the rotation at the joint. See Figure 1.



a) angular velocity

b) linear velocity

FIGURE 1. Angular and linear velocities induced by joint axis rotation.

The desired angular and linear velocities are computed from the difference between the current configuration of the end effector and the desired configuration. The angular and linear velocities of the end effector induced by the rotation of a specific joint axis are determined by the computations shown in Figure 1. The problem is to determine the best linear combination of velocities induced by the various joints that would result in the desired velocities of the end effector. By posing the problem in matrix form, the Jacobian is formed.

In assembling the Jacobian, it is important to make sure that all of the coordinate values are in the same coordinate system. It is often the case that joint specific information is given in the coordinate system local to that joint. In forming the Jacobian matrix, this information must be converted into some common coordinate system such as the global inertial coordinate system or the end effector coordinate system. Various methods have been developed for computing the Jacobian based on attaining maximum computational efficiency given the required information in local coordinate systems but all methods produce the derivative matrix in a common coordinate system.

A Simple Example

Consider the simple three revolute joint, planar manipulator of Figure 2. In this example the objective is to move the end effector, E, to the goal position, G. The orientation of the end effector is of no concern in this example.

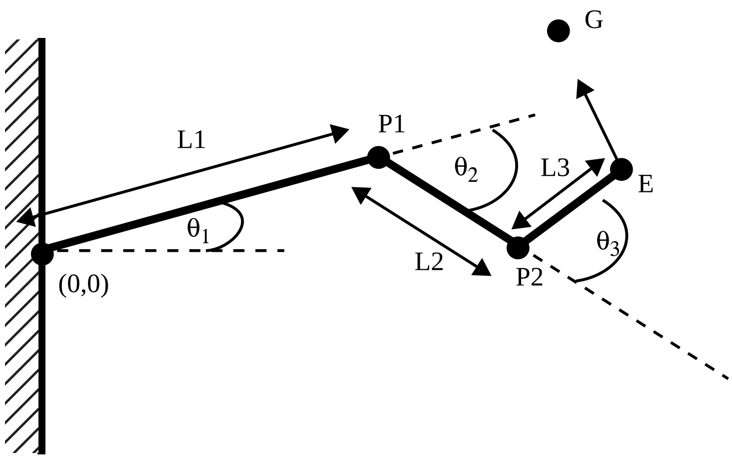


FIGURE 2. Planar, Three Joint Manipulator.

The axis of rotation of each joint is perpendicular to the figure, coming out of the paper. The effect of an incremental rotation, g_i, of each joint can be determined by the cross product of the joint axis and the vector from the joint to the end effector, V_i (Figure 3). Notice that the magnitude of the g_i's is a function of the distance between the locations of the joint and the end effector.

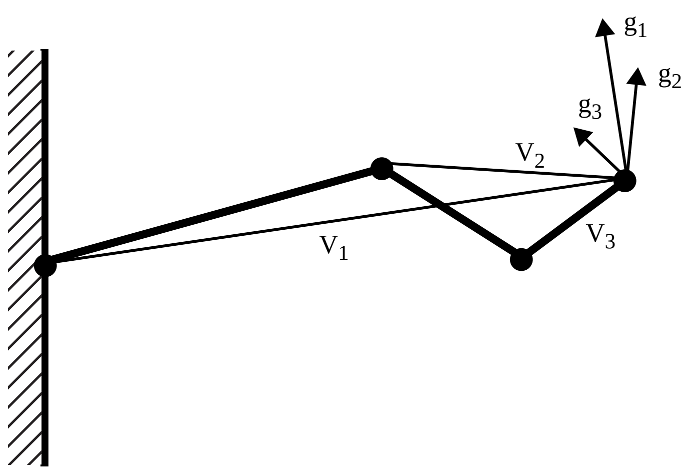


FIGURE 3. Instantaneous changes in position induced by joint angle rotations.

The desired change to the end effector is the difference between the current position of the end effector and the goal position. A vector of the desired change values is set equal to the Jacobian matrix multiplied by a vector of the unknown values: the change to the joint angles (EQ 10).

$$\begin{bmatrix}(G-E)_x\\(G-E)_y\\(G-E)_z\end{bmatrix} = \begin{bmatrix}((0,0,1)\times E)_x & (0,0,1)\times(E-P1)_x & (0,0,1)\times(E-P2)_x\\((0,0,1)\times E)_y & (0,0,1)\times(E-P1)_y & (0,0,1)\times(E-P2)_y\\((0,0,1)\times E)_z & (0,0,1)\times(E-P1)_z & (0,0,1)\times(E-P2)_z\end{bmatrix} \cdot \begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\\\dot{\theta}_3\end{bmatrix} \quad \textbf{(EQ 10)}$$

Solution Using the Inverse Jacobian

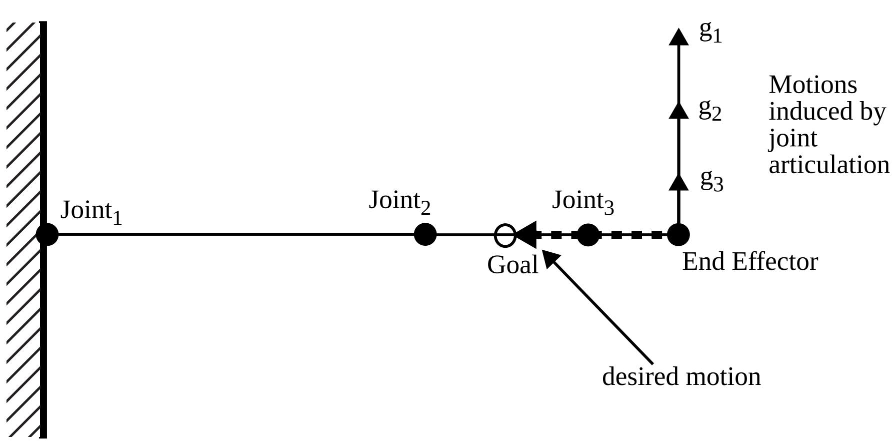
Once the Jacobian has been computed, then an equation in the form of EQ 11 must be solved.

$$V = J\dot{\theta} \quad \textbf{(EQ 11)}$$

In the case that J is square, the inverse of the Jacobian is needed to compute the joint angle velocities given the end effector velocities.

$$J^{-1}\dot{V} = \dot{\theta} \quad \textbf{(EQ 12)}$$

If the inverse of the Jacobian (J^{-1}) does not exist, then the system is said to be singular for the given joint angles. A singularity occurs when a linear combination of the joint angle velocities cannot be formed to produce the desired end effector velocities. As a simple example of such a situation, consider a fully extended, planar arm with a goal position somewhere on the forearm. In such a case, a change in each joint angle would produce a vector perpendicular to the desired direction. Obviously, no linear combination of these vectors could produce the desired motion vector. Unfortunately, determining all of the singularities of a system cannot be determined simply by visually inspecting the possible geometric configurations of the linkage.



Problems with singularities can be reduced if the manipulator is redundant - when there are more degrees of freedom than there are constraints to be satisfied. In this case, the Jacobian is not a square matrix and there are an infinite number of solutions to the inverse kinematics problem. Because the Jacobian is not square, a conventional inverse does not exist. Instead, the *pseudo inverse*, J^+, can be used (EQ 13).

$$\begin{aligned} V &= J\dot{\theta} \\ J^T V &= J^T J\dot{\theta} \\ (J^T J)^{-1} J^T \dot{V} &= (J^T J)^{-1} J^T J\dot{\theta} \\ J^+ V &= \dot{\theta} \end{aligned} \quad \text{(EQ 13)}$$

EQ 13 works because a matrix multiplied by its own transpose will be a square n by n matrix.

$J^+ = (J^T J)^{-1} J^T = J^T (J J^T)^{-1}$ is called the *pseudo-inverse* of J. It maps the desired velocities of the end effector to the required velocities of the joint angles.

$$\begin{aligned} J^+ V &= \dot{\theta} \\ J^T (J J^T)^{-1} V &= \dot{\theta} \\ \beta &= (J J^T)^{-1} V \end{aligned} \quad \text{(EQ 14)}$$

$$(J J^T)\beta = V \quad \text{(EQ 15)}$$

$$J^T \beta = \dot{\theta} \quad \text{(EQ 16)}$$

Gaussian elimination can be used to solve EQ 15 for β. This can then be substituted into EQ 16 to solve for $\dot{\theta}$.

The pseudo inverse solution computes one of many possible solutions. This solution minimizes joint angle rates. The configurations produced, however, do not necessarily correspond to what might be considered natural poses. In order to better control the kinematic model, a control expression can be added to the pseudo inverse Jacobian solution. The control expression is used to solve for control angle rates with certain attributes. The added solution contributes nothing to the desired end effector motion. The form for the control expression is shown in EQ 17. It is shown that expression does not add anything to the velocities (EQ 18)

$$\dot{\theta} = (J^+ J - I)z \quad \text{(EQ 17)}$$

$$
\begin{aligned}
V &= J\dot{\theta} \\
V &= J(J^{+}J - I)z \\
V &= (JJ^{+}J - J)z \\
V &= (J - J)z \\
V &= 0 \cdot z \\
V &= 0
\end{aligned}
\quad \text{(EQ 18)}
$$

As a consequence, the control expression can be combined with the pseudo inverse Jacobian solution so that the given velocities are still satisfied.

In order to bias the solution toward specific joint angles, H is defined as in EQ 19 where θ_i are the current joint angles, θ_{ci} are the desired joint angles, α_i are the desired angle gains, and ψ is the ψ th norm (for ψ even). z is equal to the gradient of H, ∇H (EQ 20).

$$
H = \sum_{i=1}^{n} \alpha_i \cdot (\theta_i - \theta_{ci})^{\psi} \quad \text{(EQ 19)}
$$

$$
z = \nabla_{\theta} H = \frac{dH}{d\theta} = \psi \sum_{i=1}^{n} \alpha_i \cdot (\theta_i - \theta_{ci})^{\psi - 1} \quad \text{(EQ 20)}
$$

The desired angles and gains are input parameters. The gain indicates the relative importance of the associated desired angle. The higher the gain, the stiffer the joint. If the gain for a particular joint is high, then the solution will be such that the joint angle quickly approaches the desired joint angle.

The control expression is added to the solution indicated by the conventional pseudo inverse of the Jacobian (EQ 21). If all gains are zero, then the solution will reduce to the conventional pseudo inverse of the Jacobian.

$$
\dot{\theta} = J^{+}V + (J^{+}J - I)\nabla_{\theta} H \quad \text{(EQ 21)}
$$

EQ 21 can be solved by rearranging terms as shown in EQ 22.

$$\dot{\theta} = J^{+}V + (J^{+}J - I)\nabla_{\theta}H$$
$$\dot{\theta} = J^{+}V + J^{+}J\nabla_{\theta}H - I\nabla_{\theta}H$$
$$\dot{\theta} = J^{+}(V + J\nabla_{\theta}H) - \nabla_{\theta}H \qquad \textbf{(EQ 22)}$$
$$\dot{\theta} = J^{T}(JJ^{T})^{-1}(V + J\nabla_{\theta}H) - \nabla_{\theta}H$$
$$\dot{\theta} = J^{T}[(JJ^{T})^{-1}(V + J\nabla_{\theta}H)] - \nabla_{\theta}H$$

In order to solve this, set $\beta = (JJ^{T})^{-1}(V + J\nabla_{\theta}H)$ so that EQ 22 becomes EQ 23. Use Gaussian elimination to solve for β in EQ 24. Substitute the solution for β into EQ 23 to solve for $\dot{\theta}$

$$\dot{\theta} = J^{T}\beta - \nabla_{\theta}H \qquad \textbf{(EQ 23)}$$

$$V + J\nabla_{\theta}H = (JJ^{T})\beta \qquad \textbf{(EQ 24)}$$

Simple Euler integration can be used at this point to update the joint angles. At the next time step, the Jacobian has changed so the computation must be preformed again and another step taken. This process repeats until the end effector reaches the goal configuration within some acceptable (i.e., user defined) tolerance.