18.152 - Introduction to PDEs, Fall 2004 Prof. Gigliola Staffilani

Lecture 5 - Distributions, Continued

- Convergence of Distributions

  Change the definition of $\mathcal{D}$ by replacing "differentiable" with "differentiable of any order".

  We say that a sequence of distributions f_n converges to a distribution f if

$$(f_n, \phi) \to_{\text{in } \mathbb{R}} (f, \phi)$$

  for all $\phi \in \mathcal{D}$.

- Fact: If $f_n \to f$ then $f_n' \to f'$
- Proof:

$$(f_n', \phi) = -(f_n, \phi') \to -(f, \phi') \equiv (f', \phi)$$

- Example:

  Consider the function $\chi_a(x) = \begin{cases} \frac{1}{2}a & -a < x < a \\ 0 & |x| > a \end{cases}$.

  χ_a is a distribution:

$$\begin{aligned}(\chi_a, \phi) &= \int_{\mathbb{R}} \chi_a(x)\phi(x)dx \\ &= \frac{1}{2}a \int_{|x|<a} \phi(x)dx.\end{aligned}$$

  What is $\lim_{a\to 0} \chi_a$?

$$\lim_{a\to 0}(\chi_a, \phi) = \lim_{a\to 0} \frac{1}{2a} \int_{|x|<a} \phi(x)dx$$

  Since ϕ is differentiable,

$$\lim_{a\to 0} \frac{1}{2a} \int_{|x|<a} \phi(x)dx = \phi(a).$$

  In fact,

$$\begin{aligned}\left[\frac{1}{2a}\int_{|x|<a}\phi(x)dx\right] - \phi(a) &= \frac{1}{2a}\int_{|x|<a}\left[\phi(x) - \phi(a)\right]dx \\ &= \frac{1}{2a}\int_{|x|<a}\left(\phi'(x)(x-a) + \ldots\right)dx, \text{a} \to 0 \\ &\xrightarrow[a\to 0]{} 0.\end{aligned}$$

  So $\lim_{a\to 0} \chi_a = \delta_a$.

- Definition: Support of a distribution: Let $f \in \mathcal{D}'$. Let $A = \{x| (\exists B(x,r)|\forall \phi \in \mathcal{D}$ and $\operatorname{supp} \phi \subset B(x,r), (f\phi) = 0)\}$. Then $\operatorname{supp} f = A^c$.

  Example: $\operatorname{supp} \delta_a = \{a\}$.

  Fact: If $\operatorname{supp} f$ is compact, then we can extend f to $\mathcal{C}^1(\mathbb{R}^n) \to \mathbb{C}$. The way we extend is this: let $g \in \mathcal{D}', g \equiv 1$ on $\operatorname{supp} f$. Then for $\phi \in \mathcal{C}^1(\mathbb{R}^n)$ we define $(f, \phi) \equiv (f, g\phi)$. This is defined since $g\phi \in \mathcal{D}$.

  To prove that this definition does not depend on g, let's assume that $\tilde{g}$ is also identically 1 on $\operatorname{supp} f, \tilde{g} \in \mathcal{D}$.

$$\begin{aligned} (f, g\phi) &= (f, \tilde{g}\phi) \forall \phi \in \mathcal{C}^1(\mathbb{R}^n) \\ \Leftrightarrow (f, g\phi) - (f, \tilde{g}\phi) &= 0 \\ (f, g\phi - \tilde{g}\phi) &= 0 \\ (f, (g - \tilde{g})\phi) &= 0 \\ &= 0, \text{since } \operatorname{supp}(g - \tilde{g}) \subset (\operatorname{supp} f)^c \end{aligned}$$

- Definition: Schwartz function:

$$\begin{aligned} S(\mathbb{R}^n) = S = \{\phi \in \mathbb{C}^\infty(\mathbb{R}^n) | (*)\} \\ (*) \, \forall \alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \qquad \text{(multiindices)} \\ |x^\alpha \partial^\beta \phi| \leq C_{\alpha,\beta} \end{aligned}$$

  where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \partial^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \dots \partial_n^{\beta_n}$, and $C_{\alpha,\beta}$ is allowed to depend on α, β.

  We say that $\{\phi_n\} \in S$ converges to $\phi \in S$ iff $\forall \alpha, \beta x^\alpha \partial^\beta \phi_n \xrightarrow{x \to \infty} x^\alpha \partial^\beta \phi$ uniformly.

- Definition: A tempered distribution f, is a functional $f : S(\mathbb{R}^n) \to \mathbb{R}$ such that

  1. f is linear: $(f, \alpha\phi + \beta\psi) = \alpha(f, \phi) + \beta(f, \psi)$
  2. f is continuous: for any sequence $\{\phi_n\}$ in S such that $\phi_n \to \phi$ in S, we have that $(f, \phi_n) \xrightarrow{x \to \infty} (f, \phi)$

  The set of tempered distributions is denoted by S', $S' \subset \mathcal{D}'$.

  Remark: If f is a tempered distribution, then it is also a distribution as $\mathcal{D} \subset \mathcal{S}$ and $\phi_n \to \phi$ in $\mathcal{D}$ implies convergence in S.

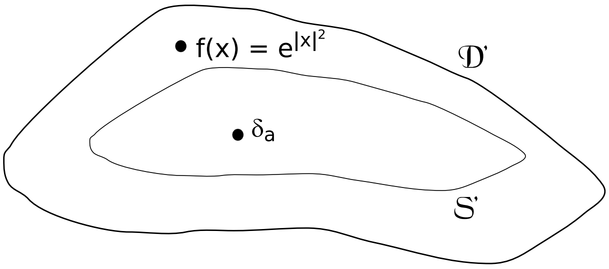
  Remark: $S' \neq \mathcal{D}'$ as there is a distribution which is not tempered:

$$(f, \phi) = \int e^{|x|^2} \phi(x) dx$$

  In fact, (f, ϕ) could be infinity because the "damping" of ϕ that decays polynomially is not strong enough to counter the exponential growth.

  If $\phi \in \mathcal{D}$ then the growth at infinity is killed by the compact support of $\phi \in \mathcal{D}$.

Remark: δ_a, for any a, is a tempered distribution.

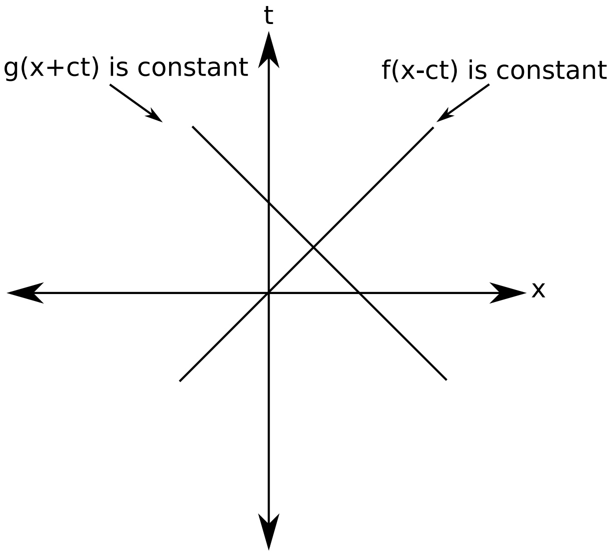


Now that we introduced some tools lets go back to the differential equations.

- The wave equation

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 \qquad \text{(leave the initial conditions unspecified for now)} \\
(\partial_t - c\partial_x)(\partial_t + c\partial_x)u &= 0 \\
\text{Change variables to } \xi = x + ct, \zeta &= x - ct \\
\partial_x u &= \partial_\xi u \frac{\partial \xi}{\partial x} + \partial_\zeta u \frac{\partial \zeta}{\partial x} \\
\partial_t u &= \partial_\xi u \frac{\partial \xi}{\partial t} + \partial_\zeta u \frac{\partial \xi}{\partial t} \\
\partial_x &= \partial_\xi + \partial_\zeta \\
\partial_t &= c(\partial_\xi - \partial_\zeta) \\
\partial_t - c\partial_x &= \cancel{c\partial_\xi} - c\partial_\zeta - \cancel{c\partial_\xi} - c\partial_\zeta = -2c\partial_\zeta \\
\partial_t + c\partial_x &= 2c\partial_\xi \\
4c^2 \partial_\zeta \partial_\xi u &= 0 \\
u_{\zeta\xi} &= 0 \\
\Rightarrow u(\xi, \zeta) &= g(\xi) + f(\zeta) \\
u(x,t) &= g(x + ct) + f(x - ct)
\end{aligned}
$$

This says that if we consider



both $g(x + ct)$ and $f(x - ct)$ are waves that travel along the lines $x + ct = \alpha, x - ct = \beta$. Assume $c > 0$, then $f(x - ct)$ travels to the right with speed c. $t = 0, x = 0$ implies a wave amplitude of $f(0)$. To see $f(0)$, we need to be at $x = ct$, which means we need to be to the right of $x = 0$.

Initial value problem:

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = \phi(x), \dot{u}(x,0) = \psi(x)$$

Now

$$u(x,t) = f(x - ct) + g(x + ct)$$
$$\dot{u}(x,t) = -cf'(x - ct) + cg'(x + ct)$$

So at time $t = 0$:

$$\begin{aligned}
f(x) + g(x) &= \phi(x) \\
-cf'(x) + cg'(x) &= \psi(x) \\
\Rightarrow f'(x) + g'(x) &= \phi'(x) \\
-f'(x) + g'(x) &= \frac{\psi(x)}{c} \\
\Rightarrow g'(x) &= \frac{1}{2}\left[\phi'(x) + \frac{\psi(x)}{c}\right] \\
f'(x) &= \frac{1}{2}\left[\phi'(x) - \frac{\psi(x)}{c}\right] \\
\Rightarrow g(x) &= \int_0^x \frac{1}{2}\left[\phi'(x') + \frac{\psi(x')}{c}\right] dx' + g(0) \\
f(x) &= \int_0^x \frac{1}{2}\left[\phi'(x') - \frac{\psi(x')}{c}\right] dx' + f(0)
\end{aligned}$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(x')dx' + A$$
$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(x')dx' + B$$
$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(x')dx' + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(x')dx'$$

Where we have used that $A + B = 0$ since $u(x,0) = \phi(x)$.

Remark: In this calculation we used that both ϕ' and ψ are continuous, but often these conditions are not satisfied:

Example:

$$u_{xx} + 2u_{xt} - 20u_{tt} = 0$$
$$Lu = 0, L = \partial_{xx} + 2\partial_{xt} - 20\partial_{tt}$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

$$a_{11} = 1, a_{12} = 1, a_{22} = -20$$
$$a_{22} - a_{12}^2 = -20 - 1 = -21 \Rightarrow \text{hyperbolic}$$
$$L = (\partial_x + \partial_t)^2 - \partial_{tt} - 20\partial_{tt}$$
$$\xi = \frac{1}{2}(x+t)$$
$$\partial_\xi = \partial_x + \partial_t$$
$$L = \partial_\xi^2 - 21\partial_t^2$$
$$u(\xi,t) = f(\xi - \sqrt{21}t) + g(\xi + \sqrt{21}t)$$
$$\Rightarrow u(x,t) = f\left(\frac{1}{2}x + \left(\frac{1}{2} - \sqrt{21}\right)t\right) + g\left(\frac{1}{2}x + \left(\frac{1}{2} + \sqrt{21}\right)t\right)$$

Definition: The source function for the wave equation is the solution to the problem

$$s_{tt} = c^2\Delta s$$
$$s(x,0) = 0, s_t(x,0) = \delta_0(x)$$

Since $\delta_0(x)$ is not a function, we need to use test functions to find s. Define

$$u(x,t) = \int s(x-y,t)\psi(y)dy$$
$$u_{tt} = \int s_{tt}(x-y,t)\psi(y)dy$$
$$c^2u_{xx} = \int s_{xx}(x-y,t)\psi(y)dy$$

thus u solves the wave equation and $u(x,0) = 0, \dot{u}(x,0) = \psi(x)$. Thus by the previous calculation we have

$$\int_{-\infty}^{\infty} s(x-y,t)\psi(y)dy = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy$$
$$\Leftrightarrow \int_{-\infty}^{\infty}\left(s(x-y,t) - \frac{1}{2c}\chi_{[x-ct,x+ct]}(y)\right)\psi(y)dy = 0$$

for all $\psi \in \mathcal{D}$, so

$$s(x,t) = \begin{cases} 0 & |x| > ct \\ \frac{1}{2c} & |x| < ct \end{cases}$$

Usually this is written using the Heaviside (or step function)

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

So $s(x,t) = H(c^2t^2 - x^2)$ for $c^2t^2 \neq x^2$.

Why is a source function useful? The solution to

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u(x,0) = 0, \dot{u}(x,0) &= \psi(x) \end{aligned}$$

can be written as

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t)\psi(y)dy$$

We already proved that u is the solution. Then

$$\begin{aligned} \lim_{t\to 0} \frac{d}{dt} \int_{-\infty}^{\infty} s(x-y,t)\psi(y)dy &= \lim_{t\to 0} \int_{-\infty}^{\infty} s_t(x-y,t)\psi(y)dy \\ &= \lim_{t\to 0}(s_t(x-y,t),\psi) \\ &= (\delta_0(x-y),\psi) \\ &= \psi(x) \end{aligned}$$

using the notion of limit of distributions.