18.152 - Introduction to PDEs, Fall 2004 Prof. Gigliola Staffilani

Lecture 4 - Types of PDEs and Distributions

- Equations of second order

Consider the up to second order case

$$0 = a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u$$

where we write $2a_{12}$ because we should have $a_{12}u_{xy} + a_{21}u_{yx}$ but $u_{xy} = u_{yx}$. Then set $2a'_{12} = a_{12} + a_{21}$ (where prime indicates the coefficients in a new equation that is equivalent to the old, then we can drop the primed indices after transforming).

The second order part of the differential operator is

$$\begin{aligned} a_{11}u_{xx} + \underbrace{2a_{12}u_{xy}}_{\text{we would like to remove this}} + a_{22}u_{yy} &= (a_{11}\partial_x^2 + 2a_{12}\partial_{xy} + a_{22}\partial_y^2)u \\ &= \left[(\partial_x + a_{12}\partial_y)^2 + (-a_{12}^2 + a_{22})\partial_y^2\right] u \qquad (1) \end{aligned}$$

by assuming $a_{11} = 1$ (which we can do without loss of generality). There are three cases based upon the sign of $a_{22} - a_{12}^2$:

1. $a_{22} - a_{12}^2 > 0$
   Set $b^2 = a_{22} - a_{12}^2$. We want to change variables such that the differential operator looks like $L = \partial_{x'}^2 + \partial_{y'}^2$ (the equation $Lu = 0$ is then known as an elliptic equation). We have

$$\begin{aligned} x &\equiv \alpha x' + \beta y' \\ y &\equiv \gamma x' + \delta y' \\ \partial_{x'}u &= u_x\frac{\partial x}{\partial x'} + u_y\frac{\partial y}{\partial x'} \\ \partial_{y'}u &= u_x\frac{\partial x}{\partial y'} + u_y\frac{\partial u}{\partial y'} \\ \Rightarrow \partial_{x'} &= \alpha\partial_x + \gamma\partial_y \\ \partial_{y'} &= \beta\partial_x + \delta\partial_y \end{aligned}$$

   This implies that in (1) we have $\alpha = 1, \beta = 0, \gamma = a_{12}, \delta = a_{22} - a_{12}^2 \Rightarrow x = x', y = a_{12}x' + (a_{22} - a_{12}^2)y'$
2. $a_{22} - a_{12}^2 < 0$
   Using the same transformation we obtain $L = \partial_{x'}^2 - \partial_{y'}^2$, and $Lu = 0$ is a hyperbolic equation.
3. $a_{22} = a_{12}^2$
   Then we obtain the operator $L = \partial_{x'}^2 +$ (lower order terms), which forms a parabolic equation.

Example: What types are the following equations?

1. $2u_{xx} + 5u_{yy} - 2u_{xy} + u_x = 0$
   We have $a_{11} = 2, a_{22} = 5, a_{12} = -2 \Rightarrow \Delta = a_{22} - a_{12}^2 = 1 \Rightarrow$ Elliptic equation
2. $u_{xx} - 6y_{xy} + 3 = 0$
   We have $a_{11} = 1, a_{22} = 0, a_{12} = -3 \Rightarrow \Delta = a_{22} - a_{12}^2 = -9 \Rightarrow$ Hyperbolic equation
3. $4u_{yy} + 4u_{xy} + u_{xx} + u_y - 1 = 0$
   We have $a_{11} = 1, a_{22} = 4, a_{12} = 2 \Rightarrow \Delta = 0 \Rightarrow$ Parabolic equation

In higher dimensions, using a similar idea of change of variables, we have the following:

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0 \tag{2}$$

here we assume $a_{ij} = a_{ji}$. If we consider the matrix $A = (a_{ij})$ this means that the matrix is symmetric.

Definition:

(2) is elliptic iff A (or $-A$) is positive definite, that is, all the eigenvalues of A (or $-A$) are greater than zero.

(2) is hyperbolic iff A (or $-A$) has eigenvalues that are all positive except one.

(2) is parabolic iff A has one zero eigenvalue and the rest are of the same sign.

In two dimensions we have

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}^2 = 0$$

$$\lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}^2 = 0$$

$$\lambda = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2)}\right]$$

Idea of the proof: We change variables like in 2D. We set

$$x' = Bx$$

$$x'_k = \sum_m b_{km} x_m$$

$$\frac{\partial}{\partial x_i} = \sum_k \frac{\partial x'_k}{\partial x_i} \frac{\partial}{\partial x'_k}$$

$$u_{x_i x_j} = \left(\sum_k b_{ki} \frac{\partial}{\partial x'_k}\right)\left(\sum_l b_{lj} \frac{\partial}{\partial x'_l}\right)$$

$$\sum_{i,j} a_{ij} u_{x_i} u_{x_j} = \sum_{k,l} \underbrace{\left(\sum_{i,j} b_{ki} a_{ij} b_{lj}\right)}_{d_{kl}} u_{x'_k} u_{x'_l}$$

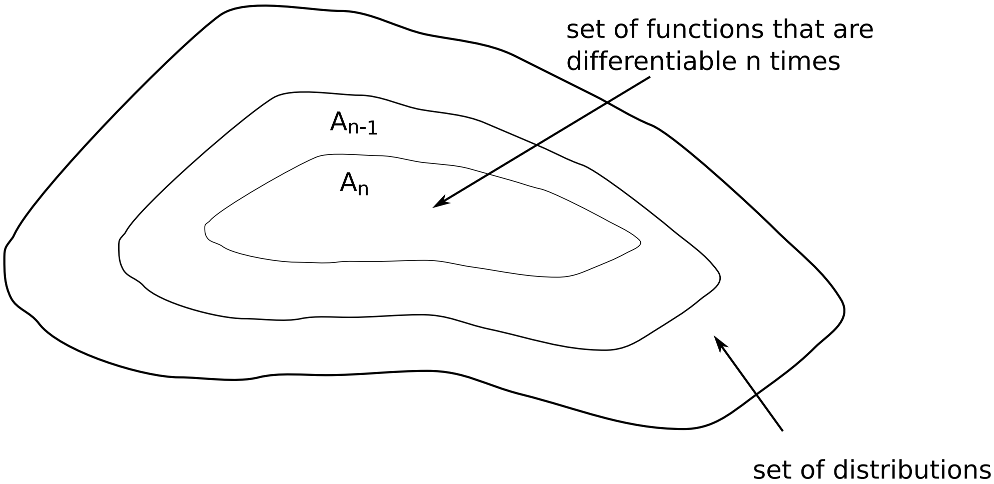
$$BAB^T = D$$

We can find B such that D is diagonal and moreover we can normalize it such that $d_{ii} = \pm 1$.

$$
\text{In two dimensions, if} \quad \begin{array}{lcl} \Delta = a_{11}a_{22} - a_{12}^2 = 0 & \Rightarrow & \lambda_1 = 0, \lambda_2 \neq 0 \\ \Delta > 0 & \Rightarrow & \lambda_1, \lambda_2 \text{ have the same sign} \\ \Delta < 0 & \Rightarrow & \lambda_1, \lambda_2 \text{ have different signs} \end{array}
$$

- Distributions:

  When we want to solve a PDE often what we want to do is this:



Distributions are not functions in general. They are, in fact, functionals, that is, operators that associate to each function a number. In this set we are allowed to perform many more "operations" than the ones we could perform in the set C^n of functions with n derivatives. For example, in the set C^{10} we cannot take the 11th derivative. Thus we work in the set of distributions, then we get regularity from the equation and we go back to the set we originally had.

Before defining distributions we define the set of "test functions" $\mathcal{D}$

$$
\mathcal{D} : \{\phi : \mathbb{R}^n \to \mathbb{R} \Big| \phi \text{ differentiable}, \exists M_\phi > 0 \text{ s.t. } \phi|_{|x|>M_\phi} \equiv 0\}
$$

Definition: A distribution is a rule (functional) $f : \mathcal{D} \to \mathbb{R}$ that is linear and continuous.

Linear:

$$
\begin{aligned} f(a\phi + b\psi) &:= (f, a\phi + b\psi) \\ &= a(f, \phi) + b(f, \psi) \end{aligned}
$$

for any $a, b \in \mathbb{R}$, and any $\phi, \psi \in \mathcal{D}$.

Continuous: If $\{\phi_n\}$ is a sequence of test functions, and there exists $M > 0$ such that

$$
\forall n \in \mathbb{N}, \phi_n \Big|_{|x|>M} = 0
$$

$$
\text{and } \phi_n \to \phi \text{ (uniformly, together with all the derivatives)}
$$

then $\lim_{n\to\infty} f(\phi_n) \equiv \lim_{n\to\infty} (f, \phi_n) = (f, \phi) = f(\phi)$.

We call $\mathcal{D}'$ the set of distributions.

- Examples:

1. Let f be integrable in $\mathbb{R}^n$. Then $f\phi$ is integrable for all $\phi \in \mathcal{D}$ and if we take

$$f(\phi) = (f, \phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$$

then f is a distribution.

2. $\delta_{x_0} : \mathcal{D} \to \mathbb{R}$
$\delta_{x_0}(\phi) = (\delta_{x_0}, \phi) = \phi(x_0)$
With an abuse of notation one also writes $\phi(x_0) = (\delta_{x_0}, \phi) = \int_{\mathbb{R}^n} \delta_{x_0}(x)\phi(x)dx$

3. Prove that $\delta_{x_0} : \mathcal{D} \to \mathbb{R}$ is a distribution.
Linearity: $\delta_{x_0}(a\phi + b\psi) = (a\phi + b\psi)(x_0) = a\phi(x_0) + b\phi(x_0) = a\delta_{x_0}(\phi) + b\delta_{x_0}(\psi)$
Continuity: $\delta_{x_0}(\phi_n) = \phi_n(x_0) \xrightarrow[n\to\infty]{\mathbb{R}} \phi(x_0) = \delta_{x_0}(\phi)$ because uniform convergence implies pointwise convergence.

4. Let f be any distribution. Verify that the functional f' defined by

$$(f', \phi) \equiv -(f, \phi')$$

satisfies linearity and continuity, which implies that f' is a distribution:

$$\begin{aligned} (f', (a\phi + b\psi)) &\equiv -(f, a\phi' + b\psi') \\ &= -a(f, \phi') - b(f, \psi') \\ &\equiv a(f', \phi) + b(f', \psi). \end{aligned}$$

Assume that $\phi_n^{(k)} \to \phi^{(k)}$ uniformly:

$$\begin{aligned} \lim_{n\to\infty} (f', \phi_n) &\equiv \lim_{n\to\infty} -(f, \phi_n) \\ &= -\lim_{n\to\infty} (f, \phi_n') \\ &= -(f, \phi') \\ &\equiv (f', \phi). \end{aligned}$$

Definition: For any distribution f, one can define its derivative f' by the previous example, which is also a distribution.