18.152 - Introduction to PDEs, Fall 2004 Prof. Gigliola Staffilani

Lecture 3 - Initial and Boundary Value Problems

- Well Posed Problems

  Physical problems have in general three characteristics which should be reflected in the mathematical equations we use to model them:

  1. Existence - The phenomenon exists $\Rightarrow$ there exists a solution to the differential equation
  2. Uniqueness - Physical processes are causal: given the state at some time we should be able to produce only one state at all later times
  3. Stability - Small changes in the initial conditions should lead to small changes in the output.

- Examples of well posed problems:

  1.
  $$\Delta u = 0$$
  $$u\Big|_{\partial\Omega} = f(x)$$

  2.
  $$\frac{\partial u}{\partial t} = \Delta u$$
  $$u(x,0) = g(x)$$

  The heat equation smooths things out.

  By contrast, the backwards heat equation,
  $$\frac{\partial u}{\partial t} = -\Delta u$$
  $$u(x,0) = f(x)$$

  is ill posed.

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$$u = \sum_n a_n(t)\sin(nx)$$
$$\Delta u = \sum_n -n^2 a_n t \sin(nx)$$
$$\frac{\partial u}{\partial t} = \sum_n \dot{a_n}(t)\sin(nx) \text{ so}$$
$$\dot{a_n}(t) = -n^2 a_n(t) \Rightarrow \frac{\partial u}{\partial t} = \Delta u$$
$$a_n(t) = e^{-n^2 t} a_n(0)$$

so for the heat equation high frequencies decay much faster than low frequencies. For the backward heat equation

$$\frac{\partial u}{\partial t} = -\Delta u,$$

we instead have

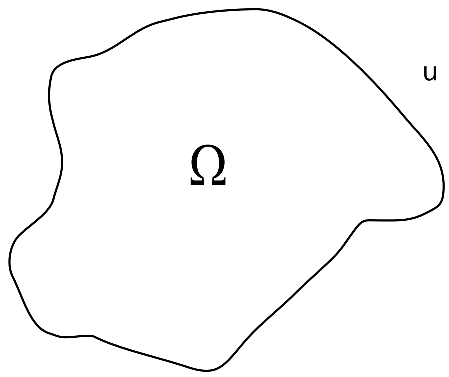
$$a_n(t) = e^{n^2 t} a_n(0).$$

Here, higher frequencies blow up.

- Initial and Boundary Conditions

  Initial conditions:

  - $u(x, t_0) = \phi(x)$
  - $u(x, t_0) = \phi(x)$, $\frac{\partial u}{\partial x}(x, t_0) = \psi(x)$



  Dirichlet conditions if $u\Big|_{\partial\Omega}$ is specified

  Neumann conditions if $\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$ is specified

  Robin conditions if $a\frac{\partial u}{\partial n} + bu\Big|_{\partial\Omega}$ is specified

- Solutions as minimizers:

$$E(u) = \int_\Omega |\nabla u|^2 = \int_\Omega \left(\left|\frac{\partial u}{\partial x^1}\right|^2 + \left|\frac{\partial u}{\partial x^2}\right|^2 + \left|\frac{\partial u}{\partial x^3}\right|^2 \right)$$

  Minimize energy:

$$\begin{aligned} 0 = \frac{dE}{d\epsilon}(u + \epsilon v)\Big|_{\epsilon=0} &= \int_\Omega \langle \nabla u, \nabla v \rangle \\ &= \int_\Omega \left(\sum_{i=1}^{3} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^i} \right) \\ &= \int_\Omega - \left(\sum_{i=1}^{3} \frac{\partial^2 u}{\partial {x^i}^2} \cdot v \right) \end{aligned}$$

$$\sum_{i=1}^{3} \frac{\partial^2 u}{\partial {x^i}^2} = 0 \Leftrightarrow \Delta u = 0$$

Suppose u is free in $\partial\Omega$. Then the divergence theorem gives that

$$\begin{aligned} \int \nabla \cdot (v \nabla u) &= \int \langle \nabla u, \nabla v \rangle + \int v \Delta u \\ \text{hence } \int \langle \nabla u, \nabla v \rangle &= \int_{\delta\Omega} v \frac{\partial u}{\partial n} - \int (\Delta u) v \end{aligned}$$

where the Neumann condition is now used.

- Conditions at infinity

  1. Schrödinger equation - $\int_{\mathbb{R}^3} |u|^2 = 1$
  2. Laplace equation - $\int |\nabla u|^2 = 1$
  3. Wave equation

     $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + Vu$$

     $\sin(\omega(t \pm x))$ are solutions to the $V \equiv 0$ case.