18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 4

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 44.1 By the maximum and minimum principles the max/min of $u(x,t) = 1 - x^2 - 2kt$ lie at either $x = 0$, or at $x = 1$, or at $t = 0$. Since $u(0,t) = 1 - 2kt$, $u(1,t) = -2kt$, $u(x,0) = 1 - x^2$, the maxima/minima at these boundaries are:
at $x = 0$: max = 1; min = 1-2kT
at $x = 1$: max = 0; min = -2kT
at $t = 0$: max = 1; min = 0.

Comparing these gives that the maximum of u over the rectangle is at $x = 0, t = 0$ (in which case $u = 1$) while the minimum is at $x = 1, t = T$ (in which case $u = -2kT$).

Problem 44.8 We have the following conditions: $u_t = ku_{xx}, (x,t) \in (0,1) \times (0,\infty)$
$u_x(0,t) = a_0 u(0,t), a_0 > 0$
$u_x(1,t) = -a_1 u(1,t), a_l > 0$

Multiplying the PDE by u and integrating from 0 to l gives

$$\int_0^l u u_t dx = k \int_0^l u u_{xx} dx.$$

But $u u_t = \frac{1}{2}\frac{\partial}{\partial t}(u^2)$, so integrating by parts on the right hand side gives

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^l u^2 dx\right] = kuu_x\Big|_0^l - k\int_0^l u_x^2 dx = -ka_l u(1,t) - ka_0 u(0,t)^2 - k\int_0^l u_x^2 dx.$$

where we used the boundary conditions. Note that all terms on the right hand side are ≤ 0, in particular so are the boundary terms, which thus contribute to the decrease of $\int_0^l u^2 dx$.

Problem 50.1

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| < l \\ 0 & \text{if } |x| > l, \end{cases}$$

so

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy = \int_{-l}^{l} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy.$$

Let $p = \frac{x-y}{\sqrt{4kt}}$, so

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\frac{x+1}{\sqrt{4kt}}}^{\frac{x-1}{\sqrt{4kt}}} e^{-p^2} dp = \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{4kt}}}^{\frac{x+1}{\sqrt{4kt}}} e^{-p^2} dp.$$

But this gives

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[\int_0^{\frac{x+1}{\sqrt{4kt}}} e^{-p^2} dp - \int_0^{\frac{x-1}{\sqrt{4kt}}} e^{-p^2} dp\right] = \frac{1}{2}\left[\mathrm{Erf}\left(\frac{x+1}{\sqrt{4kt}}\right) - \mathrm{Erf}\left(\frac{x-1}{\sqrt{4kt}}\right)\right].$$

Problem 50.4

$$\varphi(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0, \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} e^{-y} dy.$$

Combining the exponents gives

$$\begin{aligned} -\frac{(x-y)^2 + 4kty}{4kt} &= -\frac{y^2 + 2(-x+2kt)y + x^2}{4kt} \\ &= -\frac{(y+2kt-x)^2 - 4k^2t^2 + 4ktx}{4kt} \\ &= -\frac{(y+2kt-x)^2}{4kt} + kt - x, \end{aligned}$$

so

$$u(x,t) = -\frac{(y+2kt-x)^2}{4kt} e^{kt-x} \int_0^\infty e^{-\frac{(y+2kt-x)^2}{4kt}} dy.$$

Let $p = \frac{y+2kt-x}{\sqrt{4kt}}$ and change variables in the integral to get

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^k t - x \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-p^2} dp = \frac{1}{2} e^{kt-x} \left(1 - \mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}} \right) \right),$$

where we used the fact that $\frac{2}{\sqrt{\pi}} \int_0^{\frac{2kt-x}{\sqrt{4kt}}} e^{-p^2} dp = \mathrm{Erf}\left(\frac{2kt-x}{\sqrt{4kt}} \right)$.

Problem 50.16 $u_t - ku_{xx} + bu = 0, (x,t) \in \mathbb{R} \times (0,\infty)$.
$u(x,0) = \varphi(x)$.
Multiply the PDE by e^{bt} and notice that

$$e^{bt} u_t + b e^{bt} u = \frac{\partial}{\partial t}(e^{bt} u),$$

while $e^{bt} u_{xx} = \partial_x^2 (e^{bt} u)$ since e^{bt} is independent of x. Thus, $\nu = e^{bt} u$ solves

$$\nu_t - k\nu_{xx} = 0$$

$$\nu(x,0) = u(x,0) = \varphi(x)$$

(as $e^{b \cdot 0} = 1$). Hence,

$$\nu(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy,$$

and thus

$$u(x,t) = e^{-bt} \nu(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy.$$