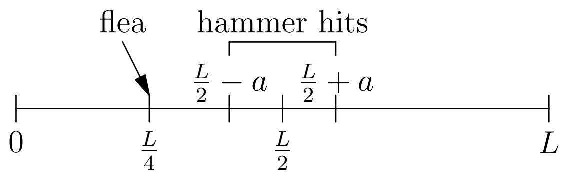
18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 3

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 36.3



From the picture we see that the flea is distance $\left(\frac{L}{2}-a\right)-\frac{1}{4}=\frac{1}{4}-a>0$ from the closest point of the hammer when it hits the string. Since waves propagate at speed $c=\sqrt{\frac{1}{\rho}}$, they reach the flea at time

$$\frac{\left(\frac{1}{4}-a\right)}{c}=\frac{\sqrt{\rho}}{T}\left(\frac{1}{4}-a\right).$$

Problem 40.1 Let $E=\frac{1}{2}\int_{-\infty}^{\infty}(u_t^2+c^2u_x^2)dx$, so $E=E(t)$ is actually a constant, independent of time, by the conservation of energy.

If u solves the wave equation with $\varphi\equiv 0$, $\psi\equiv 0$, then

$$E=E(0)=\frac{1}{2}\int_{-\infty}^{\infty}[\psi^2+c^2(\varphi')^2]dx=0,$$

so for all t we have

$$\frac{1}{2}\int_{-\infty}^{\infty}(u_t^2+c^2u_k^2)dx=E(0)=0.$$

Thus, the integrand being non-negative, we conclude that it is identically 0, so $u_t(x,t)=u_x(x,t)=0$ for all x,t. Thus u is independent of x and t. Since it is 0 at $t=0$, we conclude that $u\equiv 0$ indeed.

Problem 40.5 $u_{tt}-c^2u_0+ru_t=0, r>0$.
Multiply by u_t to integrate with respect to x.

$$\frac{d}{dt}\int\frac{1}{2}(u_t)^2dx-c^2\int u_tu_{xx}dx+r\int(u_t)^2dx=0,$$

where we used $\frac{\partial}{\partial t}(u_t)^2=2u_t\frac{\partial u_t}{\partial t}=2u_tu_{tt}$. Integrating by parts in the middle term and using $u_tu_x\Big|_{-\infty}^{+\infty}=0$ (if $u\to 0$ as $\|x\|\to\infty$ as usual, which is especially clear if u vanishes for large $\|x\|$), gives

$$\frac{d}{dt}\int\frac{1}{2}(u_t)^2dx+c^2\frac{d}{dt}\int\frac{1}{2}(u_x)^2dx+r\int u_t^2dx=0.$$

So

$$\frac{d}{dt}\int\frac{1}{2}\left[(u_t)^2+c^2(u_x)^2\right]dx=-r\int u_t^2dx\leq 0.$$

So $E=E(t)=\int\frac{1}{2}\left[(u_t)^2+c^2(u_x)^2\right]dx$ is decreasing with increasing t (i.e. it is a decreasing function of t).