18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 2

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 31.1
a) Since $u_{xy} = u_{yx}$, the PDE is of the form $u_{xx} - 4u_{xy} + u_{yy}+$(at most first order terms) $= 0$. Since

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix},$$

$\det A = 1 - 4 < 0$, so the PDE is hyperbolic.

b) $9u_{xx} + 6u_{xy} + u_{yy}+$ (lower order terms) $= 0$.

$$A = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \qquad \det A = 9 - 9 = 0.$$

So the PDE is parabolic.

Problem 31.2 $(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0.$

$$A = \begin{bmatrix} 1+x & xy \\ xy & -y^2 \end{bmatrix}.$$

So $\det A = -y^2(1 + x) - x^2y^2 = -y^2(x^2 + x + 1) = -y^2 \left(\left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \right).$

Problem 319.3 Let $L : D' \to D'$ be the derivative, so for $f \in D'$, $(Lf)(\varphi) = f'(\varphi), \forall \varphi \in D$. We need to check that for all $a, b \in \mathbb{R}$, $f, g \in D'$,

$$L(af + bg) = aLf + bLg$$

in D', i.e. for all $\varphi \in D$,

$$(L(af + bg), \varphi) = a(Lf, \varphi) + b(Lg, \varphi).$$

But by definition of L,

$$(L(af + bg), \varphi) = -(af + bg, \varphi') = -a(f, \varphi') - b(g, \varphi') = aLf(\varphi) + b(Lg)(\varphi).$$

Indeed, so L is linear.

Problem 319.8 Let $u : D \to \mathbb{R}$ be given by

$$u(\varphi) = \int\int_S \varphi dS \qquad \forall \varphi \in D.$$

Then u is linear since integration is. Moreover,

$$|u(\varphi)| = \left| \int\int_S \varphi dS \right| \le \int\int_S |\varphi| dS \le (\sup|\varphi|) \int\int_S dS = (\sup|\varphi|)(\text{Area of S}).$$

But then the argument at the end of 319.1 shows that if $\varphi_n \to \varphi$ in D, then $u(\varphi_n) \to u(\varphi)$, so u is indeed a distribution.