- If ~~$a_n, b_n > 0$~~ $a_n \geq 0$, $b_n > 0$ and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L < \infty \text{ and}$$

$\sum_{k=1}^{\infty} b_n < \infty$, then $\sum_{k=1}^{\infty} a_n$ converges.

$\boxed{\text{Lecture \# 19}}$

<u>Series of Functions</u>: $\sum_{k=1}^{\infty} f_k(x)$.

a) <u>Pointwise Convergence</u>: $\sum_{k=1}^{\infty} f_k(x) = f(x)$ pointwise on (a,b)

if for every x in (a,b)

$$\lim_{n \to \infty} \sum_{k=1}^{n} f_k(x) = f(x)$$

That is: for every x in (a,b), for every $\varepsilon > 0$

there exists $N = N(x, \varepsilon)$ s.t. if $n \geq N$, then

$$\left| \sum_{k=1}^{n} f_k(x) - f(x) \right| < \varepsilon$$

b) <u>uniform convergence</u>: as above but $N = N(\varepsilon)$ does not depend on x. In other words

$$\sum_{k=1}^{n} f_k(x) \rightrightarrows f(x)$$

For every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ s.t.

$$\max_{a \leq x \leq b} \left| \sum_{k=1}^{n} f_k(x) - f(x) \right| < \varepsilon$$

<u>Comparison test</u>: If $|f_n(x)| \le c_n$ for all x in $[a,b]$ and for all $n \ge N_0$ and $\sum_k c_k < \infty$, then

$\sum_{k=1}^{\infty} f_k(x)$ converges uniformly in $[a,b]$ and absolutely

Convergence theorem

If $f_n(x)$ are continuous on $[a,b]$ and

$\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly, then $f(x)$ is cont.

Moreover

$$\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx = \int_a^b f(x)\,dx$$

[term by term integration].

Convergence of Derivatives

If $f_n(x)$ are differentiable in $[a,b]$, $\sum_{n=1}^{\infty} f_n(c)$ converges for some c and $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly

then
$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{and}$$
$$\sum_{n=1}^{\infty} f_n'(x) = f'(x).$$

Consider the eigenvalue equation

$$X'' + \lambda X = 0 \qquad \text{on } (a,b)$$

symmetric boundary conditions

$$\left(\text{i.e. } f'(x)\overline{g(x)} - f(x)\overline{g'(x)} \Big|_a^b = 0 \right)$$

Theorem 1: There exists an infinite number of real eigenvalues λ_n, $n = 1, 2, \ldots$ s.t.

$$\lambda_n \xrightarrow[n\to\infty]{} +\infty$$

If we consider the eigenfunctions then we have the followings

a) X_n, X_m eigenfunctions ~~s.t.~~ with eigenvalues $\lambda_n \neq \lambda_m$

then $\langle X_n, X_m \rangle = 0 \Rightarrow X_n \perp X_m$!

and w.l.o.g we can assume they are real?

b) ~~X_n, X_m have same eigenvalue~~

$X_{n_1}, X_{n_2}, \ldots, X_{n_k}$ same eigenvalue λ_n

then if they are linearly independent

$$\left(\sum_{i=1}^{k} a_i X_{n_i} = 0 \iff a_i = 0 \quad i = 1, \ldots, k \right)$$

Then they can be made orthogonal (Gram-Schmidt orthogonalization procedure, Ex 10 p. 119*) and real so we can assume, ~~that ... the set~~ may be by repeating the eigenvalues more than once, that we have the sequence of eigenvalues

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \cdots \to \infty$$

and the corresponding sequence of real and orthogonal eigenfunctions

$$X_1, X_2, X_3, \cdots, X_n \cdots$$

Then we want to compare

$$\boxed{f(x)} \quad \text{with} \quad \boxed{\sum_{n=1}^{\infty} A_n X_n}$$

where
$$A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\int_a^b f(x) X_n(x)\,dx}{\int_a^b X_n^2(x)\,dx}$$

<u>Three Notations for Convergence</u> in $[a,b]$

a) $\sum_{n=1}^{\infty} f_n(x) = f(x)$ pointwise

$\Leftrightarrow \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \xrightarrow[N \to \infty]{} 0$

b) $\sum_{n=1} f_n(x) = f(x)$ uniformly

* Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York: Wiley, 3 March 1992. ISBN: 0471548685.

if $\sup_{a\le x\le b}\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \xrightarrow[N\to\infty]{} 0$ (5)

c) $\sum_{n=1}^{\infty} f_n(x) = f(x)$ in mean square (or L^2)

iff

$$\left\| \sum_{n=1}^{N} f_n(x) - f(x) \right\|_{L^2} \xrightarrow[N\to\infty]{} 0$$

$$\Updownarrow$$

$$\int_a^b \left| \sum_{n=1}^{N} f_n(x) - f(x) \right|^2 dx$$

Remark 1) If $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly

then it is also true in L^2

Proof:

$$\int_a^b \left| \sum_{n=1}^{N} f_n(x) - f(x) \right|^2 dx \le \max_{a\le x\le b} \left| \sum_{n=1}^{N} f_n(x) - f(x) \right| (b-a) \xrightarrow[N\to\infty]{} 0\cdot(b-a) = 0$$

Remark 2) uniform convergence

$\Longrightarrow$ point wise convergence

Remark 3

L^2 convergence $\not\Rightarrow$ pointwise convergence

and hence $\not\Rightarrow$ uniform convergence

~~Example~~

$$g_n(x) = \begin{cases} 1 & \left[\frac{1}{4} - \frac{1}{n^2}, \frac{1}{4} + \frac{1}{n^2}\right] \quad n \text{ odd} \\ 1 & \left[\frac{3}{4} - \frac{1}{n^2}, \frac{3}{4} + \frac{1}{n^2}\right] \quad n \text{ even} \\ 0 & \text{other } x \end{cases}$$

$$\int |g_n(x) - 0|^2 dx = \int g_n(x)^2 dx = \int g_n(x) dx$$

$$-\tfrac{3}{4} \quad 0 \quad \tfrac{1}{4} \quad \tfrac{3}{4} \quad \tfrac{7}{4}$$

n odd $$\int_{-\frac{3}{4}}^{\frac{7}{4}} g_n(x) = \int_{\frac{1}{4}-\frac{1}{n^2}}^{\frac{1}{4}+\frac{1}{n^2}} dx = \frac{1}{4} + \frac{1}{n^2} - \frac{1}{4} + \frac{1}{n^2} = \frac{2}{n^2}$$

n even $$\int_{-\frac{3}{4}}^{7/4} = \int_{\frac{3}{4}-\frac{1}{n^2}}^{\frac{3}{4}+\frac{1}{n^2}} dx = \frac{2}{n^2} \xrightarrow[n\to\infty]{} 0$$

Pointwise

take $x = \frac{1}{4}$ then $g_n(\tfrac{1}{4}) = 1$ for all n odd

$g_n(\tfrac{1}{4}) = 0$ for all n even

$\Rightarrow$ no convergence

Remark 4: a) pointwise convergence $\not\Rightarrow$ uniform convergence

b) pointwise convergence $\not\Rightarrow$ L^2 convergence

(homework: ex 3 page 129)[*].

Ex: $f_n(x) = \dfrac{nx}{1+n^2x^4} - \dfrac{(n-1)x}{1+(n-1)^2x^4}$ in $0<x<1$

This is a telescopic series:

~~Definition: $\sum_{n=1}^{\infty} a_n$ is telescopic iff~~

in the sense that

~~$\sum_{n=1}^{m} a_n = a_m - a_n$~~

$$\sum_{n=1}^{N} a_n = a_N$$

In fact

$$f_1(x) = \frac{x}{1+x^4} - 0$$

$$f_2(x) = \frac{2x}{1+4x^4} - \frac{x}{1+x^4} \quad - - -$$

$$\sum_{n=1}^{N} = \frac{Nx}{1+N^2x^4}$$ for any $0<x<1$

$$\lim_{N\to\infty} \frac{Nx}{1+N^2x} = 0$$

[*] Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York: Wiley, 3 March 1992. ISBN: 0471548685.