MATH 152: THE FOURIER TRANSFORM – THE INVERSION FORMULA

Recall that $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions, i.e. the functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with the property that for any multiindices $\alpha, \beta \in \mathbb{N}^n$, $x^\alpha \partial^\beta \phi$ is bounded. Here we wrote $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and $\partial^\beta = \partial_{x_1}^{\beta_1} \ldots \partial_{x_n}^{\beta_n}$; with $\partial_{x_j} = \frac{\partial}{\partial x_j}$. (This notation with α, β, is called the multiindex notation.)

We defined the Fourier transform on $\mathcal{S}$ as

$$(\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, \phi(x)\, dx, \tag{1}$$

and the inverse Fourier transform as

$$(\mathcal{F}^{-1}\psi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \psi(\xi)\, d\xi. \tag{2}$$

We showed by integration by parts that $\mathcal{F}, \mathcal{F}^{-1}$ satisfy

$$\mathcal{F} D_{x_j} \phi = \xi_j \mathcal{F}\phi,\ -D_{\xi_j}\mathcal{F}\phi = \mathcal{F}(x_j\phi),\ D_{x_j} = i^{-1}\partial_j, \tag{3}$$

with similar formulae for the inverse Fourier transform:

$$\mathcal{F}^{-1} D_{\xi_j}\psi = -x_j \mathcal{F}\psi,\ D_{x_j}\mathcal{F}^{-1}\psi = \mathcal{F}^{-1}(\xi_j\psi). \tag{4}$$

We used this to show that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ and similarly for $\mathcal{F}^{-1}$; indeed, if $\phi \in \mathcal{S}$, then $x^\alpha \partial^\beta \phi$ is bounded for all multiindices α, β. But the Fourier transform of this a constant multiple of $\partial^\alpha \xi^\beta \hat{\phi}$. But we in fact have that $(1+|x|^2)^{(n+1)/2} x^\alpha \partial^\beta \phi$ is also bounded (the first factor in effect simply increases α), so $|x^\alpha \partial^\beta \phi| \leq C(1+|x|^2)^{-(n+1)/2}$ for some $C > 0$. Thus,

$$\begin{aligned} |\partial^\alpha \xi^\beta \hat{\phi}(\xi)| &= |\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, (x^\alpha \partial^\beta \phi)(x)\, dx| \\ &\leq \int_{\mathbb{R}^n} |e^{-ix\cdot\xi}\, (x^\alpha \partial^\beta \phi)(x)|\, dx \leq \int_{\mathbb{R}^n} C(1+|x|^2)^{-(n+1)/2} = M < +\infty, \end{aligned}$$

so $\sup |\partial^\alpha \xi^\beta \hat{\phi}| \leq M$, i.e. $\partial^\alpha \xi^\beta \phi$ is bounded indeed. Although the derivatives and the multiplications are in the opposite order as in the definition of $\mathcal{S}$, using Leibniz' rule (i.e. the product rule) for differentiation, we get other terms of the same form, so we conclude that $\hat{\phi} \in \mathcal{S}$ indeed. The proof for the inverse Foruier transform is of course very similar.

We also calculated the Fourier transform of the Gaussian $\phi(x) = e^{-a|x|^2}$, $a > 0$, on $\mathbb{R}^n$ (note that $\phi \in \mathcal{S}$!) by writing it as

$$\hat{\phi}(\xi) = \left(\int_{\mathbb{R}} e^{-ax_1^2}\, dx_1 \right) \ldots \left(\int_{\mathbb{R}} e^{-ax_n^2}\, dx_n \right),$$

hence reducing it to one-dimensional integrals which can be calculated by a change of variable and shift of contours. We can also proceed as follows. Write x for the one-dimensional variable, ξ for its Fourier transform variable for simplicity, and $\psi(x) = e^{-ax^2}$,

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-ax^2}\, dx = e^{-\xi^2/4a} \int_{\mathbb{R}} e^{-a(x+i\xi/(2a))^2}\, dx,$$

where we simply completed the square. We wish to show that

$$f(\xi) = \int_{\mathbb{R}} e^{-a(x+i\xi/(2a))^2}\, dx$$

is a constant, i.e. is independent of ξ, and in fact it is equal to $\sqrt{\pi/a}$. But that is easy: differentiating f, we obtain $f'(\xi) = -i\int_{\mathbb{R}}(x+i\xi/(2a))e^{-a(x+i\xi/(2a))^2}\,dx$. The integrand is the derivative of $(-1/(2a))e^{-a(x+i\xi/(2a))^2}$ with respect to x, so by the fundamental theorem of calculus, $f'(\xi) = (i/(2a))e^{-a(x+i\xi/(2a))^2}|_{x=-\infty}^{+\infty} = 0$, due to the rapid decay of the Gaussian at infinity. This says that f is a constant, so for all ξ, $f(\xi) = f(0) = \int_{\mathbb{R}} e^{-ax^2}\,dx$ which can be evaluated by the usual polar coordinate trick, giving $\sqrt{\pi/a}$. Returning to $\mathbb{R}^n$, the final result is thus that

$$\hat{\phi}(\xi) = (\pi/a)^{n/2}e^{-|\xi|^2/4a},$$

which is hence another Gaussian. A similar calculation shows that for such Gaussians $\mathcal{F}^{-1}\hat{\phi} = \phi$, i.e. for such Gaussians $T = \mathcal{F}^{-1}\mathcal{F}$ is the identity map.

Now we can show that T is the identity map on all Schwartz functions using the following lemma.

Lemma 0.1. *Suppose $T : \mathcal{S} \to \mathcal{S}$ is linear, and commutes with x_j and D_{x_j}. Then T is a scalar multiple of the identity map, i.e. there exists $c \in \mathbb{C}$ such that $Tf = cf$ for all $f \in \mathcal{S}$.*

Proof. Let $y \in \mathbb{R}^n$. We show first that if $\phi(y) = 0$ and $\phi \in \mathcal{S}$ then $(T\phi)(y) = 0$. Indeed, we can write, essentially by Taylor's theorem, $\phi(x) = \sum_{j=1}^n (x_j - y_j)\phi_j(x)$, with $\phi_j \in \mathcal{S}$ for all j. In one dimension this is just a statement that if ϕ is Schwartz and $\phi(y) = 0$, then $\phi_1(x) = \phi(x)/(x-y) = (\phi(x) - \phi(y))/(x-y)$ is Schwartz: smoothness near y follows from Taylor's theorem, while the rapid decay with all derivatives from $\phi_1(x) = \phi(x)/(x-y)$. For the multi-dimensional version, one can take $\phi_j(x) = (x_j - y_j)\phi(x)/|x-y|^2$ for $|x-y| \geq 2$, say, suitably modified inside this ball. Thus,

$$T\phi = \sum_{j=1}^n (x_j - y_j)(T\phi_j),$$

where we used that T is linear and commutes with multiplication by x_j for all j. Substituting in $x = y$ yields $(T\phi)(y) = 0$ indeed.

Thus, fix $y \in \mathbb{R}^n$, and some $g \in \mathcal{S}$ such that $g(y) = 1$. Let $c(y) = (Tg)(y)$. We claim that for $f \in \mathcal{S}$, $(Tf)(y) = c(y)f(y)$. Indeed, let $\phi(x) = f(x) - f(y)g(x)$, so $\phi(y) = f(y) - f(y)g(y) = 0$. Thus, $0 = (T\phi)(y) = (Tf)(y) - f(y)(Tg)(y) = (Tf)(y) - c(y)f(y)$, proving our claim.

We have thus shown that there exists $c : \mathbb{R}^n \to \mathbb{C}$ such that for all $f \in \mathcal{S}$, $y \in \mathbb{R}^n$, $(Tf)(y) = c(y)f(y)$, i.e. $Tf = cf$. Taking $f \in \mathcal{S}$ such that f never vanishes, e.g. a Gaussian as above, shows that $c = Tf/f$ is $\mathcal{C}^\infty$, since Tf and f are such.

We have not used that T commutes with D_{x_j} so far. But

$$\begin{aligned} c(y)(D_{x_j}f)(y) = T(D_{x_j}f)(y) = D_{x_j}(Tf)|_{x=y} &= D_{x_j}(c(x)f(x))|_{x=y} \\ &= (D_{x_j}c)(y)f(y) + c(y)(D_{x_j}f)(y). \end{aligned}$$

Comparing the two sides, and taking f such that f never vanishes, yields $(D_{x_j}c)(y) = 0$ for all y and for all j. Since all partial derivatives of c vanish, c is a constant, proving the lemma. □

The actual value of c can be calculated by applying T to a single Schwartz function, e.g. a Gaussian, and then the explicit calculation from above shows that $c = 1$, so $\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}$ indeed.