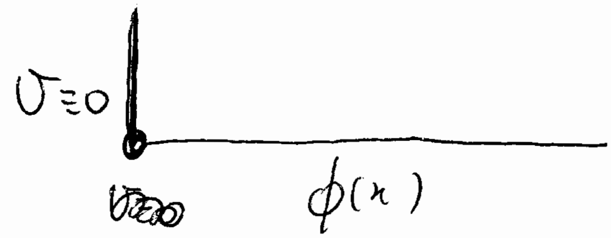
Lecture #12 Diffusion on Half line

Consider the diffusion problem

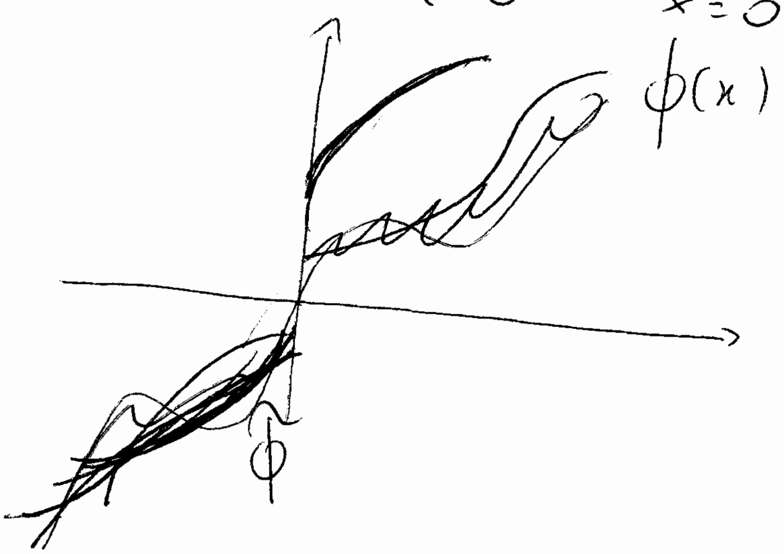
$$(D_+)\begin{cases} v_t - k v_{xx} = 0 & x>0,\ t>0 \\ v(x,0) = \phi(x) & t=0,\ x>0 \\ v(0,t) = 0 & x=0,\ t>0 \end{cases}$$



To solve (D_+) we want to extend ϕ also to negative values of x then solve the problem for all x in $\mathbb{R}$.

Extension of ϕ

We define $\tilde{\phi}(x) = \begin{cases} \phi(x) & x>0 \\ -\phi(-x) & \text{if } x<0 \\ 0 & x=0 \end{cases}$



We notice that $\tilde{\phi}$ is an odd function.

$$u(-x,t) = \frac{1}{\sqrt{}} \int_{-\infty}^{\infty} e^{-(-x-y)^2/4kt} \phi(y)\,dy$$ ③

change variable $y = -\xi$

$$= +\frac{1}{\sqrt{}} \int_{-\infty}^{\infty} e^{-(-x+\xi)^2/4kt} \phi(-\xi)\,d\xi$$

$$= \frac{1}{\sqrt{}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2} -\phi(\xi)\,d\xi$$

$$= -u(x,t).$$

So u odd means that

$$u(0,t) = 0 \implies v(0,t) = 0 \checkmark$$

More precisely

$$u(x,t) = \int_{0}^{\infty} S(x-y,t)\,\phi(y)\,dy + \int_{-\infty}^{0} S(x-y,t)\,\phi(-y)\,dy$$

$$= \int_{0}^{\infty} [S(x-y,t) - S(x+y,t)]\,\phi(y)\,dy$$

This method is called $\boxed{\text{reflection method}}$

Remark: Suppose we have the more general problem

$$\begin{cases} v_t - k\,v_{xx} = 0 \\ v(x,0) = \phi(x) \\ v(0,t) = f(t) \end{cases}$$

~~[illegible]~~ Definition:

$\psi(x)$ is an odd function iff

$$\psi(-x) = -\psi(x) \implies \psi(0) = 0$$

Then we look at the problem

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \tilde{\phi}(x) \\ \cancel{u(0,t)} \end{cases}$$

We solve this so that

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, \tilde{\phi}(y)\, dy \qquad t>0,\ x \text{ in } \mathbb{R}$$

~~[illegible]~~. Let $v(x,t) = u(x,t)$ for $x>0$

Fact: $v(x,t)$ is the solution for the problem on half line

To see this follows the following steps

i) u solves equation for all x so in particular it solves it for $x>0 \implies v$ is sol. to equation

ii) Because $\tilde{\phi}$ is odd then also u is odd. To see this

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy$$

$$v(\ell,t) = \frac{1}{2}\tilde\phi(\ell+ct) + \frac{1}{2}\tilde\phi(\ell-ct) + \frac{1}{2c}\int_{\ell-ct}^{\ell+ct}\tilde\psi(s)\,ds$$ (10)

$$\frac{1}{2}\tilde\phi(\ell+ct) - \frac{1}{2}\tilde\phi(ct-\ell) + \frac{1}{2c}\int_{-ct}^{ct}\tilde\psi(\xi+\ell)\,d\xi$$

$\xi = s-\ell$ $\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$ $\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$ see picture.

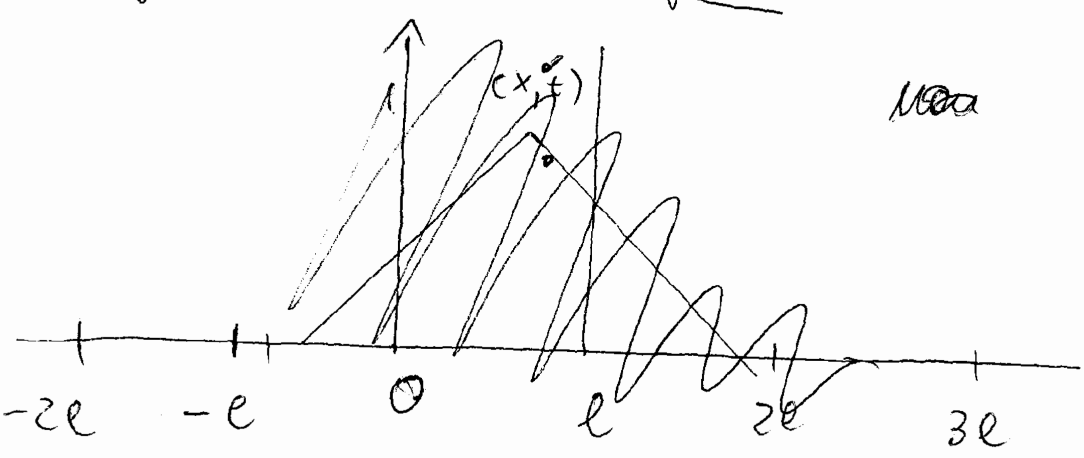
Now check that $\tilde\phi(-\ell+ct) = \tilde\phi(-\ell+ct+2\ell) = \tilde\phi(ct+\ell)$

Finally $t=0$

$$v(x,t) = \frac{1}{2}\tilde\phi(x) + \frac{1}{2}\tilde\phi(x) + 0 \quad \text{in } (0,\ell)$$

$$= \frac{2}{2}\phi(x) = \phi(x).$$

<u>Making the formula more explicit</u>



$$\int_{\ell}^{2\ell} -\psi(2\ell-s)\,ds + \int_{2\ell}^{3\ell}\psi(2\ell+s)\,ds + \int_{3\ell}^{x+ct} -\psi(4\ell-s)\,ds.$$

now let's change variables

$$+\int_{2\ell-x+ct}^{0}\psi(\xi)\,d\xi \cancel{+\int_{0}^{\ell}\psi(\xi)\,d\xi} \cancel{-\int_{0}^{\ell}\psi(\xi)\,d\xi} + \int_{0}^{\ell}\psi(s)\,ds$$

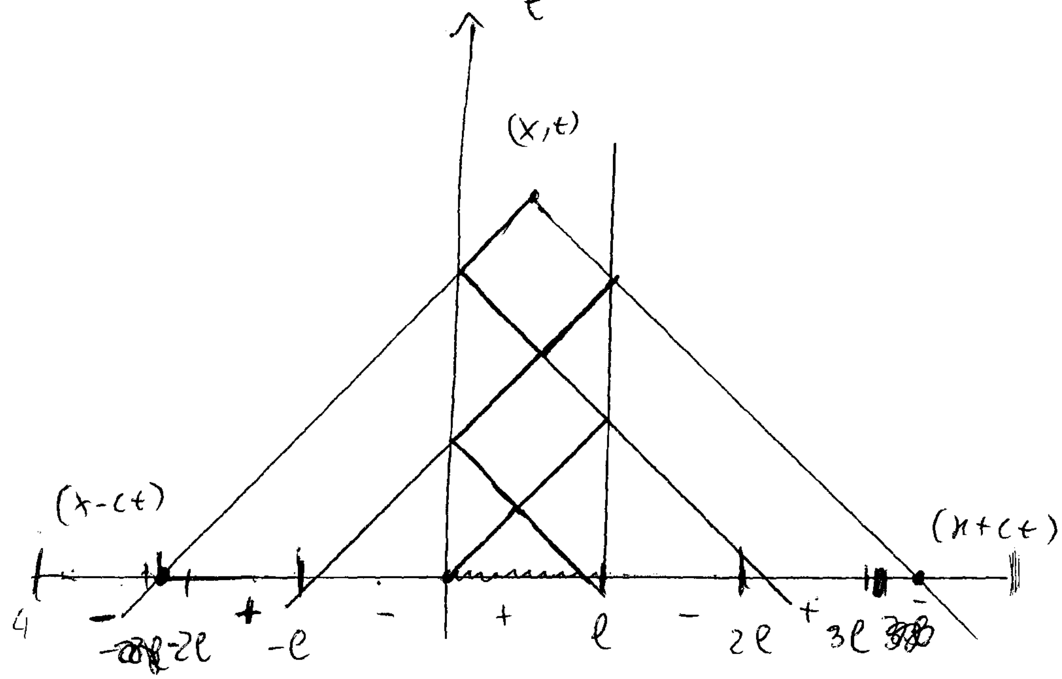
$$\cancel{-\int_{0}^{\ell}\psi(\xi)\,d\xi} \cancel{+\int_{0}^{\ell}\psi(\xi)\,d\xi} + \int_{\ell}^{4\ell-x-ct}\psi(\xi)\,d\xi$$

$$= \int_{2\ell-x+ct}^{4\ell-x-ct}\psi(\xi)\,d\xi$$

So the formula becomes

$$\boxed{v(x,t) = -\frac{1}{2}\phi(4\ell-x-ct) - \frac{1}{2}\phi(2\ell-x+ct) + \frac{1}{2c}\int_{2\ell-x+ct}^{4\ell-x-ct}\psi(\xi)\,d\xi}$$

Q.E.D.



First of all

$$\tilde{\phi}(x+ct) = -\phi(4l - x - ct)$$

$$\tilde{\phi}(x-ct) = -\phi(2l - x + ct)$$

Now we have to analyze the integral

$$\int_{x-ct}^{x+ct} \psi(s)\,ds = \int_{x-ct}^{-2l} + \int_{-2l}^{-l} + \int_{-l}^{0} + \int_{0}^{l} + \int_{l}^{2l} + \int_{2l}^{3l}$$

$$+ \int_{3l}^{x+ct}$$

$$\int_{x-ct}^{-2l} \tilde{\psi}(s)\,ds = \int_{x-ct}^{-2l} -\psi(2l - s)\,ds$$

$$+ \int_{-2l}^{-l} \psi(2l + s)\,ds + \int_{-l}^{0} -\psi(-s)\,ds + \int_{0}^{l} \psi(s)\,ds +$$

Assume $t \geq 0$

$$v(x,t) = \frac{1}{2}\left[\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(y)\,dy$$

if $\boxed{x > ct}$ all the arguments are > 0 so

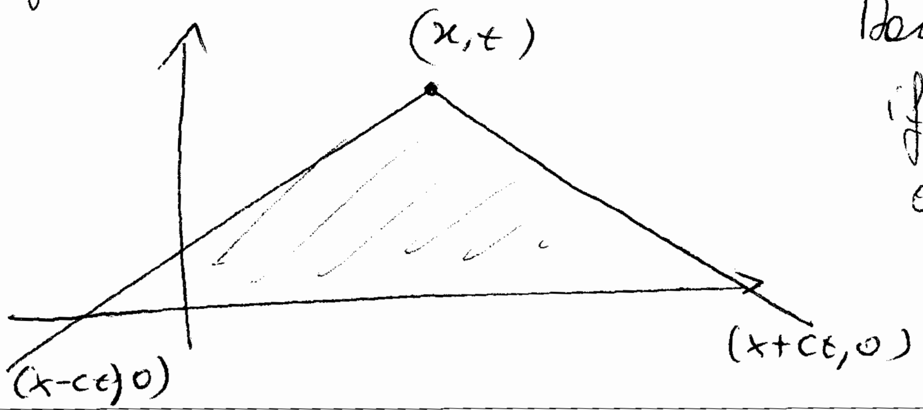
$$\boxed{v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy}$$

$\boxed{0 < x < ct}$

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy \underbrace{- \frac{1}{2c}\int_{x-ct}^{0} \psi(-y)\,dy}_{= -\frac{1}{2c}\int_{0}^{ct-x} \psi(y)\,dy}$$

$$\boxed{v(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy}$$

Interpretation of the formula in terms of domain of dependence

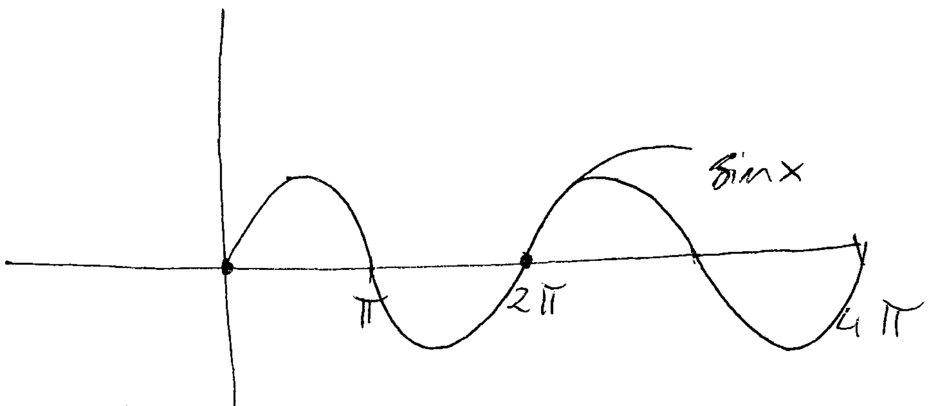


Domain of influence if we are looking at the problem on the whole line

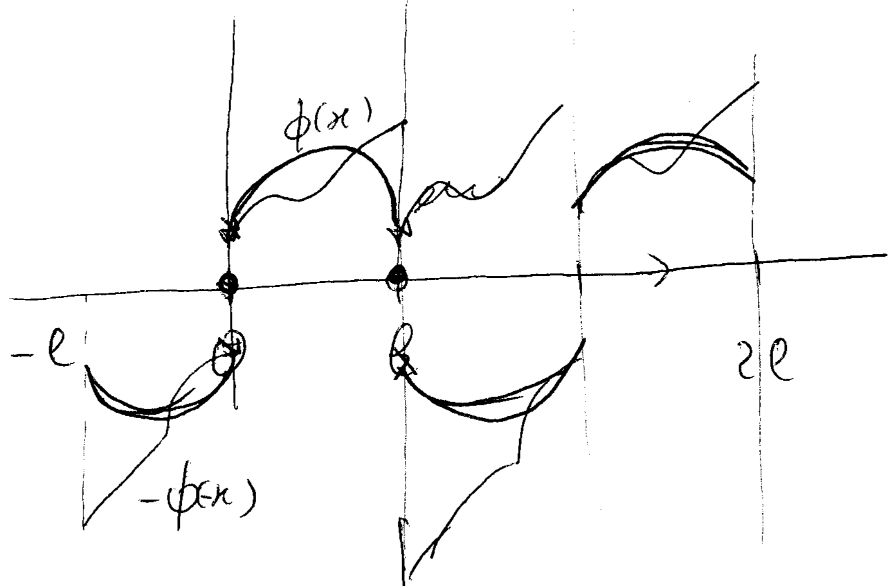
Definition: A function $f(x)$ is periodic with period p ⑧ if

$$f(x+p) = f(x)$$

Example: $f(x) = \sin x$, $g(x) = \cos x$ are functions with $p = 2\pi$



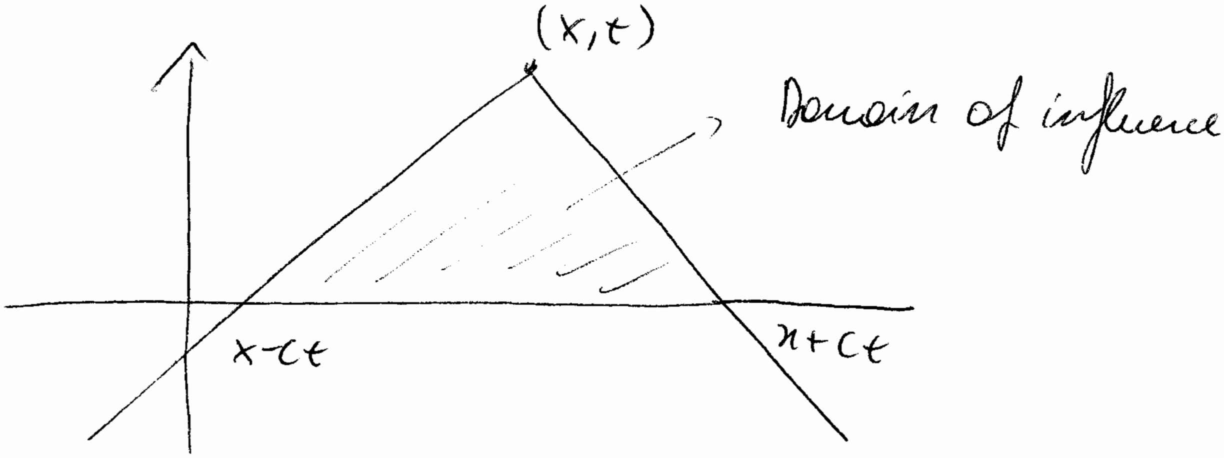
We extend ϕ and ψ using odd extension and periodicity:



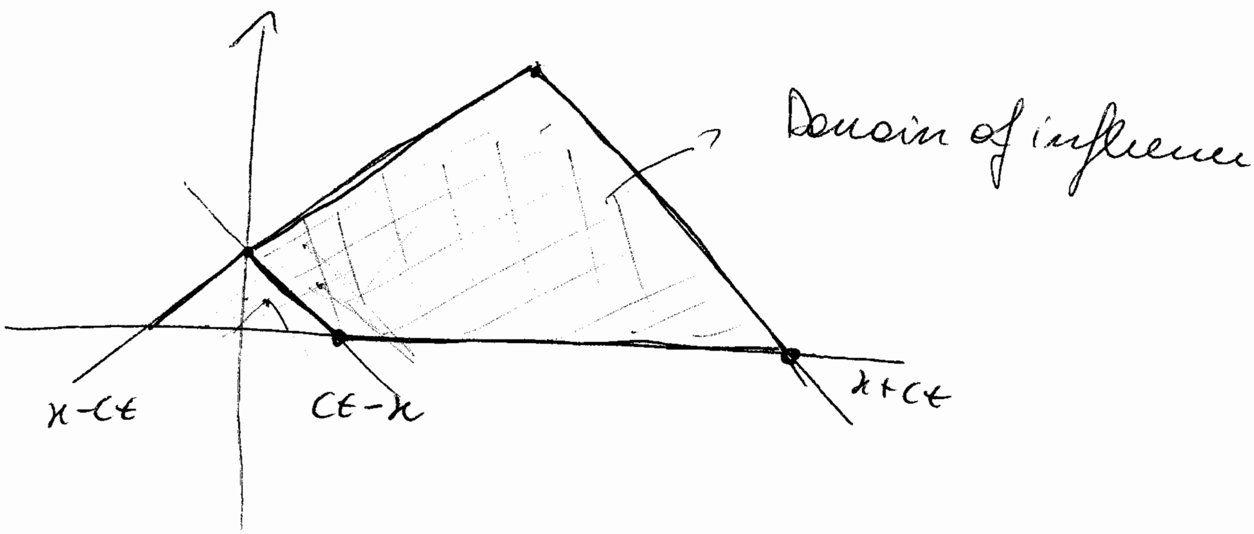
So in $[-\ell, \ell]$

$$\tilde{\phi}_e(x) = \begin{cases} \phi(x) & \text{if } x \text{ in } (0, \ell] \\ -\phi(x) & \text{if } x \text{ in } [-\ell, 0) \end{cases}$$ extend of period 2ℓ

if $x - ct > 0$ then

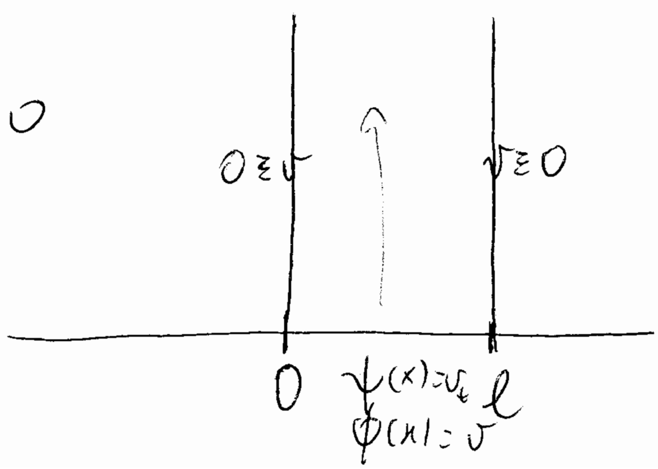


if $x - ct < 0$ and $x + ct > 0$



Consider now a finite interval

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = \phi(x) \qquad 0<x<l \\ v_t(x,0) = \psi(x) \\ v(0,t) = v(l,t) = 0 \end{cases}$$



Then the extension needed is the one of a even function i.e.

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & x > 0 \\ \phi(x) & x < 0 \\ 0 & x = 0 \end{cases}$$

This because if a function is even then its derivative is odd, hence $v_x(0,t)$ becomes automatic.

$\boxed{\text{Reflections of Waves}}$: We consider the Initial-Boundary problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & 0 < x < \infty \\ v(x,0) = \phi(x) & v_t(x,0) = \psi(x) \\ v(0,t) = 0 \end{cases}$$

We proceed like in the case of the diffusion equation by odd extension:

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases} \qquad \tilde{\psi}(x) = \begin{cases} \psi(x) & x > 0 \\ -\psi(-x) & x < 0 \\ 0 & x = 0 \end{cases}$$

We solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \tilde{\phi}(x) \quad u_t(x,0) = \tilde{\psi}(x) \end{cases}$$

Because $\tilde{\phi}, \tilde{\psi}$ are odd also u is odd (homework)

So $v = u$ for $x > 0$ is solution of the problem.

Define $w(x,t) = v(x,t) - f(t)$

Then $w(0,t) = v(0,t) - f(t) = 0$

$$w(x,0) = v(x,0) - f(0) = \phi(x) - f(0)$$

~~so letts thake~~

$$w_t = v_t - f_t$$

$$w_{xx} = v_{xx}$$

So w solves ↙ nonhogeneous problem

$$\begin{cases} w_t - k\, w_{xx} = f_t(t) \\ w(x,0) = \phi(x) - f(0) \\ w(0,t) = 0 \end{cases}$$

Assume $f_t \equiv 0 \Rightarrow f(t) = C_0$ then you solve the proble

$$w(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)][\phi(y) - C_0]\,dy$$

$$v(x,t) = w(x,t) + f(t)$$

~~Reflection of trees~~

<u>Remark</u>: If u start with Neumann condition

$$\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = \phi(x) \\ v_x(0,t) = 0 \end{cases}$$

Do the same for ψ to get $\tilde{\psi}_e$

Then we obtain a solution $u(x,t)$ for t in $\mathbb{R}$ and x in $\mathbb{R}$.

We define

$$v(x,t) = u\Big|_{[0,e]}(x,t)$$

Clearly v is a solution because u is for all (x,t)

To check the boundary and initial conditions we have

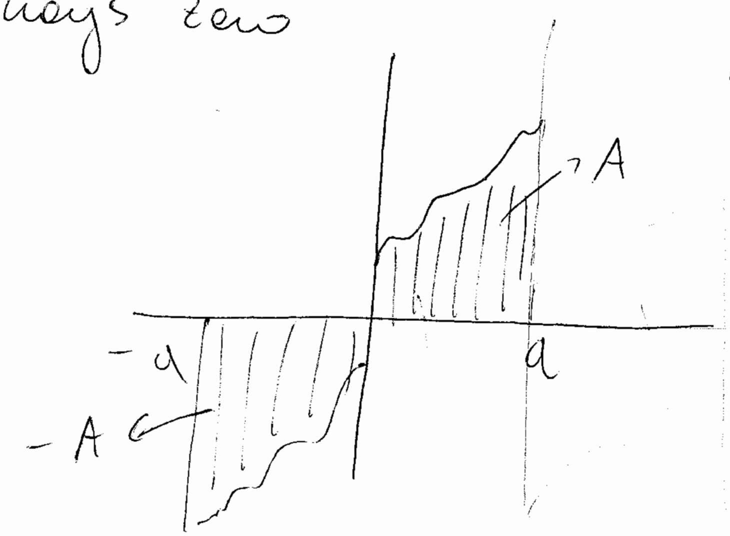
$$v(x,t) = \frac{1}{2}\tilde{\phi}(x+ct) + \frac{1}{2}\tilde{\phi}(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\tilde{\psi}(s)\,ds$$

now if $x=0$

$$v(0,t) = \frac{1}{2}\tilde{\phi}(ct) + \frac{1}{2}\tilde{\phi}(-ct) + \frac{1}{2c}\int_{-ct}^{ct}\tilde{\psi}(s)\,ds$$

$$= \cancel{\frac{1}{2}\tilde{\phi}(ct)} - \cancel{\frac{1}{2}\tilde{\phi}(ct)} + 0$$

The integral of a odd function on an interval $[-a,a]$ is always zero



$A - A = 0$