Lecture # 16

More on the Robin conditions

Recall that we considered the wave equation and the diffusion equation ~~with~~ on $[0, \ell]$ with the Robin conditions

$$\begin{cases} u_x(0,t) - a_0 u(0,t) = 0 \\ u_x(\ell,t) + a_\ell u(\ell,t) = 0 \end{cases}$$

If one looks for solutions of type

$$u(x,t) = X(x)\,T(t)$$

then the eigenvalue problem for $X(x)$ becomes

$$\begin{cases} X'' = -\lambda X \\ X' - a_0 X = 0 & x = 0 \\ X' + a_\ell X = 0 & x = \ell \end{cases}$$

λ = eigenvalues

We observed that depending on the sign of $a_0 a_\ell$ and $a_0 + a_\ell$ we get different situations for the eigenvalues. This is a summary of what we found

Case 1: $a_0, a_\ell > 0$

In this case there are only positive eigenvalues λ_n and $\frac{n^2\pi^2}{\ell^2} \leq \lambda_n < \frac{(n+1)^2\pi^2}{\ell^2}$ $\quad n = 0, 1, 2 \ldots$

Case 2 $\quad a_0 < 0, a_\ell > 0 \quad a_0 + a_\ell > 0$

subcase a: $\quad a_0 + a_\ell > -a_0 a_\ell \ell$

Only positive eigenvalues and again

$$\frac{n^2\pi^2}{\ell^2} < \lambda_n < \frac{(n+1)^2\pi^2}{\ell^2} \quad (*)$$

subcase b: $\quad a_0 + a_\ell = -a_0 a_\ell \ell$

$\lambda_0 = 0$, all the other ones are positive as in $(*)$ as above

subcase c: $\quad a_0 + a_\ell < -a_0 a_\ell \ell$

$\lambda_0 < 0$ all the rest are positive, λ_n $n \geq 1$ as in $(*)$

In Ex 8: you will describe the hyperbola

$$a_0 + a_\ell = -a_0 a_\ell \ell$$

for fixed ℓ and based on this you will get the complete picture.

The solutions $u(x,t)$ with the Robin conditions will be

$$u(x,t) = \sum_n T_n(t) X_n(x)$$

where

$X_n(x)$ has been described before and

$$T_n(x) = \begin{cases} A_n e^{-\lambda_n t} & \text{diffusion} \\ A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct) & \text{waves} \end{cases}$$

<u>Problem</u>: Solve the wave equation with Robin conditions s.t. $a_0 + a_\ell < -a_0 a_\ell \ell$

From our discussion we have

$$\lambda_0 < 0$$

$$\frac{n^2\pi^2}{\ell^2} < \lambda_n < (n+1)^2 \frac{\pi^2}{\ell^2} \qquad n = 1, 2 \ldots$$

$$T_0(x) = A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}$$

$$-\gamma_0^2 = \lambda_0$$

$$T_n(x) = A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)$$

$$X_0(x) = \cosh \gamma_0 x + \frac{a_0}{\gamma_0} \sinh \gamma_0 x$$

$$X_n(x) = \cos \sqrt{\lambda_n}\, x + \frac{a_0}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\, x$$

$$u(x,t) = \left(A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}\right)\left(\cosh \gamma_0 x + \frac{a_0}{\gamma_0} \sinh \gamma_0 x\right)$$

$$+ \sum_{n=1} \left[A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)\right]\left(\cos \sqrt{\lambda_n}\, x + \frac{a_0}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\, x\right)$$

Remark:

$$\lim_{t \to \infty} e^{\gamma_0 ct} = +\infty$$

So while the other terms of the waves keep oscillating the one involving $e^{\gamma_0 ct}$ grows. $a_e > 0$ $\quad a_0 + a_e > 0$

To fix the ideas assume $a_0 < 0$. This means that the string absorbs the energy at the point $x_0 = 0$ the condition

$$a_0 + a_e < -a_0 a_e l$$

$$-a_0(a_e l + 1) > a_e$$

$$-a_0 > \frac{al}{a_l l + 1}$$

So there is a lot more absorption of energy than release of it at l. This explains why for long time the ~~oscillation~~ amplitude of the wave solution becomes larger and larger.

Fourier Series

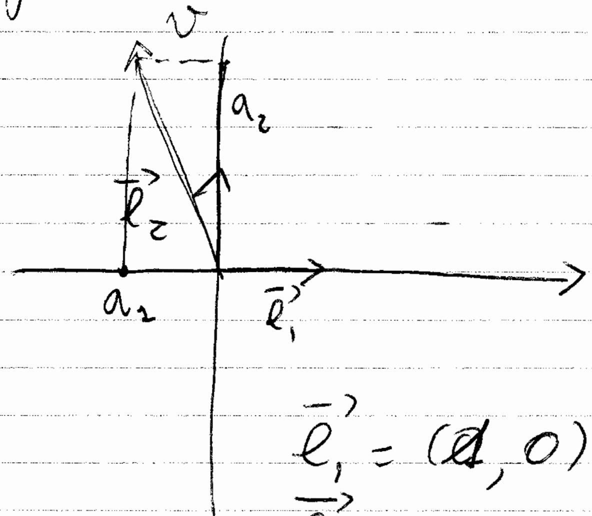
The great fact here is that any "reasonable"* function periodic of period l can be written as an infinite linear combination of sin and cos functions.

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} + B_n \cos\frac{n\pi x}{l}$$

In what sense we mean the "∞" sum will be made clear later. Once one "believes" this then the problem will be finding A_n and B_n.

Analogy:

Consider the space $\mathbb{R}^2$



$B = \{\vec{l_1}, \vec{l_2}\}$ $\quad$ $\vec{l_1} = (1, 0)$

$\vec{l_2} = (0, 1)$

Any other vector can be written as a linear combination of $\vec{l_1}, \vec{l_2}$

$$\vec{v} = a_1 \vec{l_1} + a_2 \vec{l_2}$$

Question: Given $\vec{v}$ how do we find a_1 and a_2?

a_1 = size (with sign) of projection of v on $\vec{l_1}$

a_2 = " " " " on $\vec{l_2}$

$a_1 = v \cdot l_1$

$a_2 = v \cdot l_2$

Check: $v = a_1 l_1 + a_2 l_2$ $\quad$ $v \cdot l_1 = a_1 \vec{l_1} \cdot \vec{l_1} + a_2 \underbrace{\vec{l_2} \cdot \vec{l_1}}_{=0} = a_1$

So if we imagine $\left\{ \sin \frac{n\pi}{\ell}, \cos \frac{n\pi}{\ell} \right\}$ to be a basis of vectors then A_n and B_n can be found using an appropriate "$\cdot$" product. Clearly we need the property

1) $$\sin \frac{n\pi}{\ell}x \bullet \sin \frac{m\pi}{\ell}x = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

dot product

2) $$\cos \frac{n\pi}{\ell}x \bullet \cos \frac{m\pi}{\ell}x = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

The right definition for "$\bullet$" product in our context is:

Given two periodic functions of period ℓ f and g we define

$$f \bullet g = \frac{2}{\ell} \int_0^{\ell} f(x)\, g(x)\, dx$$

check 1) and 2).

~~Consider the functions~~ $\cancel{e^{i\frac{m\pi}{\ell}x} = \cos \frac{m\pi}{\ell}x + i \sin \frac{m\pi}{\ell}x}$

1)' $$\frac{2}{\ell} \int_0^{\ell} \sin n\frac{\pi}{\ell}x \cdot \sin m\frac{\pi}{\ell}x \, dx = \delta_{nm}$$

Recall

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

$$\frac{\cancel{2}}{\ell}\int_0^{\ell} \frac{1}{\cancel{2}} \cos(n-m)\frac{\pi}{\ell}x \, dx$$

$$-\frac{\cancel{2}}{\ell}\int_0^{\ell} \frac{1}{\cancel{2}} \cos(n+m)\frac{\pi}{\ell}x \, dx$$

$$\cancel{= \frac{2}{\ell}\frac{1}{2}\frac{\ell}{\pi}\sin(n-m)\frac{\pi}{\ell}x}$$

if $n = m$

$$= \frac{\cancel{2}}{\ell}\cdot\frac{1}{\cancel{2}}\ell - \frac{\cancel{2}}{\ell}\int_0^{\ell}\frac{1}{\cancel{2}}\cos 2n\frac{\pi}{\ell}x \, dx =$$

$$2n\frac{\pi}{\ell}x = \theta$$

$$1 - \underbrace{\frac{\cancel{2}}{\ell}\int_0^{2n\pi}\frac{1}{\cancel{2}}\cos\theta \frac{\ell}{2n\pi}\, d\theta}_{\text{by periodicity } = 0} = 1$$

if $n \neq m$

$$= \frac{\ell}{\pi(n-m)}\sin(n-m)\frac{\pi}{\ell}x - \frac{1}{\pi(n+m)}\sin(n+m)\frac{\pi}{\ell}x\Big|_0^{\ell} = 0$$

Now we we expect

$$\phi(x) = \sum_{n=1}^{M} A_n \sin \frac{n\pi x}{\ell}$$

then

$$A_m = \phi(x) \cdot \sin \frac{m\pi x}{\ell}$$

to check this

$$\phi(x) \cdot \sin \frac{m\pi x}{\ell} = \frac{2}{\ell}\int_0^{\ell} \sum_{n=1}^{M} A_n \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell}$$

$$= \frac{2}{\ell} \sum A_n \underbrace{\int_0^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell}\, dx}$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ A & \text{if } n = m \end{cases}$$

So the only surviving term in the sum is A_m!

<u>Back to wave and diffusion equations</u>

Recall that for example

$$\begin{cases} u_{tt} - c u_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0 \quad \leftarrow \text{Dirichlet} \\ u_t(0) = \psi \quad u(0) = \phi \end{cases}$$

then the ~~conditions~~ solution

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi c t}{\ell} + B_n \sin \frac{n\pi c t}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

where A_n and B_m are such that

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{\ell}$$

$$\psi(x) = \sum_n B_m \frac{n\pi c}{\ell} \sin \frac{n\pi x}{\ell}$$

So based on what we found above

$$A_n = \frac{2}{\ell} \int_0^\ell \phi(x) \sin \frac{n\pi x}{\ell} \, dx$$

$$B_n = \frac{\cancel{\ell}}{n\pi c} \frac{2}{\cancel{\ell}} \int_0^\ell \psi(x) \sin \frac{n\pi x}{\ell} \, dx$$

What we did for the sin equation works also for the cos. We want to find the coefficient of the representation

$$(*) \quad \phi(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{\ell} = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{\ell}$$

One can prove with similar methods that

$$\frac{2}{\ell} \int_0^\ell \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} \, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

So

$$A_0 = \frac{2}{\ell}\int_0^\ell \cos\frac{0\pi x}{\ell}\,\phi(x)\,dx = \underbrace{\frac{2}{\ell}\int_0^\ell \phi(x)\,dx}_{2\text{ average of }\phi\text{ on }[0,\ell]}$$

So $\frac{1}{2}A_0$ = average of ϕ on $[0,\ell]$

$$A_n = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\frac{n\pi x}{\ell}\,dx$$

~~the cos representation~~ is

While the sin representation was useful for the Dirichlet problem, the cos one is useful for the Neumann problem.

Because both sin and cos provide good "bases" for the representation of a "reasonable" nice periodic function, we can combine them and get the

Full Fourier Series

Definition: assume ϕ is a function, periodic of period 2ℓ, or more simply ϕ is defined on $[-\ell, \ell]$ then

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{\ell} + B_n \sin \frac{n\pi x}{\ell} \right)$$

is the Fourier Series of ϕ.

The basis for this representation (or eigenfunctions) are now

$$\left\{ 1, \cos(n\pi x/\ell), \sin(n\pi x/\ell), n = 1, 2 \ldots \right\}$$

Fact:

$$\frac{1}{\ell} \int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} \, dx = 0 \quad \text{for all } n, m$$

$$\frac{1}{\ell} \int_{-\ell}^{\ell} \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} \, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\frac{1}{\ell} \int_{-\ell}^{\ell} \sin \frac{n\pi x}{\ell} \sin n\frac{\pi}{\ell} x \, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\frac{1}{\ell} \int_{-\ell}^{\ell} 1 \cdot \cos \frac{n\pi x}{\ell} \, dx = 0 = \int_{-\ell}^{\ell} 1 \cdot \sin \frac{n\pi x}{\ell} \, dx$$

another way to see this is to use

$$e^{i\frac{n\pi x}{\ell}} = \cos\frac{n\pi x}{\ell} + i\sin\frac{n\pi x}{\ell}$$

$$\frac{1}{2\ell}\int_{-\ell}^{\ell} e^{i\frac{n\pi x}{\ell}} \cdot e^{-i\frac{m\pi x}{\ell}}\,dx$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{i(n-m)\frac{\pi x}{\ell}}\,dx$$

if $n = m$ $\quad \overset{\shortparallel}{1}$

$n \neq m \quad \frac{1}{2\ell}\,\frac{\ell}{(n-m)\pi}\,\underbrace{e^{i(n-m)\frac{\pi x}{\ell}}\Big|_{-\ell}^{\ell}}_{=0}$

<u>Example</u>: Consider the function

$$\phi(x) = x \quad \text{on } [0,\ell]$$

Find it Fourier Series

$$A_0 = \frac{2}{\ell}\int_0^{\ell} x\,dx = \frac{1}{\ell}\,\frac{x^2}{2}\Big|_0^{\ell} = \frac{\ell}{2}$$

$$A_n = \frac{2}{\ell}\int_0^{\ell} x\left(\cos\frac{m\pi}{\ell}x\right)dx$$

$$= \frac{2}{\ell}\left[\underbrace{x\,\sin\left(\frac{n\pi}{\ell}x\right)\cdot\frac{\ell}{n\pi}\Big|_0^{\ell}}_{=0} - \frac{\ell}{n\pi}\int_0^{\ell}\sin\frac{n\pi x}{\ell}\,dx\right]$$

$$= \frac{2}{\cancel{\ell}}\,\frac{\cancel{\ell}}{n\pi}\,\frac{\ell}{n\pi}\,\cos\frac{n\pi x}{\ell}\Big|_0^{\ell} = \frac{2\ell}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ -\dfrac{4l}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$x = \frac{l}{2} - \sum_{n=1,3,\ldots} \frac{4l}{n^2\pi^2} \cos \frac{n\pi}{l} x$$

$$B_n = \frac{2}{l}\int_0^l x \sin \frac{n\pi}{l} x \, dx \quad \ldots$$