Solutions

MATH 152, FALL 2004: MIDTERM #2

Problem #1 20

a) Using Fourier Transform solve the initial value roblem with diffusion equation with variable dissipation

$$(1) \qquad \begin{cases} u_t - Ku_{xx} + bt^2 u = 0 \\ u(0,t) = \phi(x) \end{cases}$$

for $K > 0$, $-\infty < x < \infty$ and $t > 0$.

b) Write the solution u above more explicitly when $\phi(x) = e^{-x^2}$

For this second part you may need to remember that if F denotes the Fourier transform, then

$$F(e^{-x^2/2})(\xi) = (2\pi)^{1/2} e^{-\xi^2/2} \text{ and } F(f(ax))(\xi) = a^{-1}\hat{f}(\xi/a)$$

for any constant a.

Problem #2 10

Solve the initial and boundary value problem

$$(2) \qquad \begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u(x,0) = 0, \ u_t(x,0) = V \\ u_t(0,t) + au_x(0,t) = 0 \end{cases}$$

for $0 < x < \infty$, $V, a, c > 0$ and $a > c$.

Hint: Solve first the problem with $h(x,t) = V = 0$.

Problem #3 30

Consider the equation

$$(3) \qquad u_t = Ku_{xx} - \alpha u,$$

with $\alpha > 0$. This equation models a one dimensional road with heat loss through the lateral sides with zero outside temperature. Suppose the road has length L and the boundary conditions

$$(4) \qquad u(0,t) = u(L,t) = 0$$

a) The equilibrium temperatures are the functions u constant with respect to time, hence they are solutions of

$$(5) \qquad \begin{cases} u_{xx} - \alpha u = 0 \\ u(0) = u(L) = 0. \end{cases}$$

Find all the solutions $u(x)$ of (5).

b) Solve the boundary problem given by (3) and (4) with initial data $u(x,0) = f(x)$ using the method of separation of variables. Make sure you analyze ALL the eigenvalues of the problem!

c) Analyze the temperature solution $u(x,t)$ obtained in b) for large time $t \to \infty$ and compare it with what you found in part a).

Problem #4 10

Using carefully the properties of the Fourier Transform and distributions, prove that for any constant b we have

$$\hat{b}(\xi) = 2b\pi\delta(\xi) \tag{6}$$

Problem #5 30

Consider the vibrating string with fixed ends

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \;\; u_t(x,0) = 0, \end{cases} \tag{7}$$

for $0 < x < L$.

a) Using the periodic extension method prove that

$$u(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct), \tag{8}$$

where $F(x)$ is an odd periodic extension of $f(x)$.

b) Use now separation of variables to solve again (7).

c) Because F is odd and periodic, $F(x)$ admits a sin expansion, namely

$$F(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L). \tag{9}$$

Use (9) and the trigonometric identity

$$\begin{aligned} \sin a \cos b &= 1/2[\cos(a-b) - \cos(a+b)], \\ \cos a &= \sin(\pi/2 - a) \\ \sin(a \pm b) &= \sin a \cos b \pm \sin b \cos a \end{aligned}$$

to show that the two solutions obtained in a) and b) coincide.

Solutions Midterm # 2

Problem #1

1) Take F.T. of initial value problem:

$$\begin{cases} \hat{u}_t - k(i\xi)^2\hat{u} + bt^2\hat{u} = 0 \\ \hat{u}(0) = \hat{\phi}(\xi) \end{cases}$$

For fixed ξ this is an ODE

$$\begin{cases} \hat{u}_t = (-k\xi^2 + bt^2)\hat{u} = 0 \\ \hat{u}(0) = \hat{\phi}(\xi) \end{cases}$$

$$\frac{\hat{u}_t}{\hat{u}} = -k\xi^2 + bt^2$$

$$\parallel$$

$$(\ln\hat{u})' = -k\xi^2 + bt^2 \qquad \text{integrate}$$

$$\int_0^t (\ln\hat{u})'(s)\,ds = \int_0^t (-k\xi^2 + bs^2)\,ds$$

$$\ln\frac{\hat{u}(t)}{\hat{u}(0)} = -kt\xi^2 + \frac{b}{3}t^3$$

$$\hat{u}(\xi,t) = \hat{u}(0)\, e^{-kt\xi^2 + \frac{b}{3}t^3}$$

$$\hat{u}(\xi,t) = e^{-\frac{b}{3}t^3}\,\hat{\phi}(\xi)e^{-kt\xi^2}$$ 2

So

$$u(x,t) = e^{-\frac{b}{3}t^3}\,\mathcal{F}^{-1}\left(\hat{\phi}(\xi)e^{-kt\xi^2}\right)(x)$$

b) Now assume $\phi(x) = e^{-x^2}$, then

$$\phi(\xi) = e^{-(\sqrt{2}x)^2/2}$$

$$\hat{\phi}(\xi) = \mathcal{F}\left(f(ax)\right)(\xi) = a^{-1}f\left(\frac{\xi}{a}\right)$$

when $f(x) = e^{-x^2/2}$ $a=\sqrt{2}$

$$= 2^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}e^{-\frac{\xi^2}{4}}$$

$$\hat{\phi}(\xi)e^{-kt\xi^2} = \frac{1}{\sqrt{2}}(2\pi)^{\frac{1}{2}}e^{-\xi^2\left(kt+\frac{1}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}(2\pi)^{\frac{1}{2}}e^{-\frac{\left(\xi\sqrt{2}\sqrt{kt+\frac{1}{4}}\right)^2}{2}}$$

so if $b^{-1} = \sqrt{2}\sqrt{kt+\frac{1}{4}}$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}\sqrt{kt+\frac{1}{4}}}(2\pi)^{\frac{1}{2}}b^{-1}e^{-\frac{(\xi b^{-1})^2}{2}}$$

$$= \frac{1}{\sqrt{4kt+1}}\,\mathcal{F}\left(e^{-\frac{(bx)^2}{2}}\right)(\xi)$$

so $$u(x,t) = e^{-\frac{b}{3}t^3} \frac{1}{\sqrt{4kt+1}} e^{-\frac{x^2}{(4kt+1)}}$$ (3)

Problem # 2 (this is a combination of problem #6, page 64 and comments in page 76)

First define $v(x,t) = u(x,t) - tV$

then $v(x,t)$ solves ~~comments in page 76~~

$$\begin{cases} v_{tt} - c^2 v_{xx} = h(x,t) \\ v(x,0) = 0 \quad v_t(x,0) = 0 \\ v_t(0,t) + a u_x(0,t) = -V \end{cases}$$

because

$$v_t(x,t) = u_t(x,t) - V$$

$$v_x(x,t) = u_x(x,t)$$

We look for a solution like

$$v(x,t) = g(x+ct) + f(x-ct)$$

From initial conditions:

$$\begin{cases} g(x) + f(x) = 0 \qquad x \geq 0 \\ c g'(x) - c f'(x) = 0 \end{cases}$$

~~[illegible]~~ From second equation (5)

$g(x) = f(x) + c_0$ hence

$$f(x) = -g(x) = c_1$$

So if $x - ct > 0 \Rightarrow x > ct \Rightarrow v(x,t) \equiv 0$

Now we consider the case $x < ct$

$v_t(x,t) + a\, v_x(x,t) =$

$c\, g'(x+ct) - c f'(x-ct) + a\left(g'(x+ct) + f'(x-ct)\right)$

because $x + ct > 0$ (here we assume $c > 0, t > 0$)

$$g'(x+ct) = 0$$

$$f'(x-ct)\,[a-c]$$

So for $x = 0$ $\quad f'(-ct)(a-c) = -V$

$$f'(-ct) = -\frac{V}{a-c}$$

$$f'(s) = -\frac{V}{a-c} \quad \text{for } s < 0$$

$$f(s) = -\frac{V}{a-c}\, s + c_2$$

So for $x < ct$

$$v(x,t) = -\frac{V}{a-c}(x-ct) + C_3$$

Now if $h \neq 0$ then

$$v(x,t) = \begin{cases} \int_\Delta h(y,s)\,dy\,ds & x > ct \\ -\frac{V}{a-c}(x-ct) + C_3 + \int_{\tilde{\Delta}} h(y,s)\,dy\,ds & x < ct \end{cases}$$

where Δ and $\tilde{\Delta}$ are the appropriate domains of dependence.

Problem #3

a) We look for $u(x) = Ae^{\beta x}$

$$u_{xx} - \alpha u = \beta^2 u - \alpha u = 0$$

$\Leftrightarrow$ $\beta^2 = \alpha > 0$ $\quad u(x) = Ae^{\sqrt{\alpha}x}$

$u(0) = A \quad u(L) = Ae^{\sqrt{\alpha}L} = 0 \Leftrightarrow A = 0$

$$u \equiv 0$$

b) $u(x,t) = X(x)T(t)$

$$X(x)\,T'(t) = k\,X''(x)\,T(t) - \alpha\,X(x)\,T(t)$$

$$\frac{T'(t)}{T} = k\,\frac{X''(x) - \alpha X(x)}{X(x)} = -\lambda$$

$$k\,X''(x) - \alpha\,X(x) = -\lambda\,X(x)$$

$$\begin{cases} X''(x) = -\left(\dfrac{\lambda-\alpha}{k}\right) X(x) \\ X(0) = X(L) = 0 \end{cases}$$

We know that for this eigenvalue problem we only have positive eigenvalues, more precisely $\dfrac{\lambda-\alpha}{k} = \left(\dfrac{n\pi}{L}\right)^2 \quad n = 1, 2 \ldots$

hence $\lambda_n = \alpha + k\left(\dfrac{n\pi}{L}\right)^2$

and $X_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$

then $\dfrac{T'(t)}{T(t)} = -\lambda_n \qquad (\ln T(t))' = \lambda_n$

$$\ln T(t) = -\lambda_n t \; A_n$$

$$T(t) = A_n\, e^{\lambda_n t}$$

then

$$u(x,t) = \sum_{n=1} A_n e^{-\left(\alpha + k\left(\frac{n\pi}{L}\right)^2\right)t} \sin\frac{n\pi x}{L}$$

(finite sum for now)

c) $\lim_{t\to\infty} u(x,t) = 0$ due to the fact that

$$\lim_{t\to\infty} e^{-\left[\alpha + k\left(\frac{n\pi}{L}\right)\right]^2 t} = 0$$

Problem #4 : We need to use distribution notation : For any test function ϕ

$$\langle \hat{b}, \phi \rangle = \langle b, \hat{\phi} \rangle = b \langle 1, \hat{\phi} \rangle$$

$$= b \int \hat{\phi}(\xi)\, d\xi = 2\pi b \frac{1}{2\pi}\int e^{-ix\cdot\xi} \hat{\phi}(\xi)\, d\xi \Big|_{x=0}$$

$$= b\, 2\pi\, \phi(0) = b 2\pi \langle \delta, \phi \rangle$$

Hence $\langle \hat{b}, \phi \rangle = \langle 2\pi b \delta, \phi \rangle \iff$

$$\hat{b} = 2\pi b \delta$$

Problem #5

a) extend $f(x)$ to $\tilde{f}(x) = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0 \\ -f(-x) & -L < x < 0 \end{cases}$

(8)

Then

$f_{ext}(x) =$ periodic extension of $\tilde{f}$

Then we solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f_{ext}, \quad u_t(x,0) = 0 \end{cases}$$

So

$$u(x,t) = f_{ex}(x+ct) + f_{ext}(x-ct)$$

We then restrict to $0<x<L$. Clearly the BC is satisfied because the odd extension.

$$u(x,t) = \frac{1}{2}\Big(F(x+ct) + F(x-ct)\Big).$$

b) $T(t)\,X(x) = u(x,t)$

$X T''(t) = c^2 X_{xx} T$

$$\frac{T''(t)}{c^2 T} = \frac{X''}{X} = -\lambda$$

Eigenvalue problem $\begin{cases} X'' = -\lambda X \\ X(0) = X(L) = 0 \end{cases}$

$\lambda > 0 \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, \ldots$

$X_n(x) = \sin\frac{n\pi x}{L}$

$T''(t) = -c^2\left(\frac{n\pi}{L}\right)^2 T(t)$

$T(t) = A_n \cos\frac{cn\pi}{L} t$

$u(x,t) = \sum_n A_n \cos\frac{n\pi}{L}ct \sin\frac{n\pi x}{L}$

c) Now from last trig identity

$\sin(a+b) + \sin(a-b) = 2\sin a\cos b$

So if $a = \frac{n\pi x}{L}$ $\quad b = \frac{n\pi}{L}ct$

then

$$u(x,t) = \sum_n \frac{A_n}{2}\left[\sin\left\{\frac{n\pi}{L}(x+ct)\right.\right.$$

$$\left. + \sin\frac{n\pi}{L}(x-ct)\right]$$

and moreover

$u(\cancel{x},\cancel{0}) = f(x) \Rightarrow \sum_n A_n \sin\frac{n\pi x}{L} = f(x)$

hence

$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right]$ (sin a).