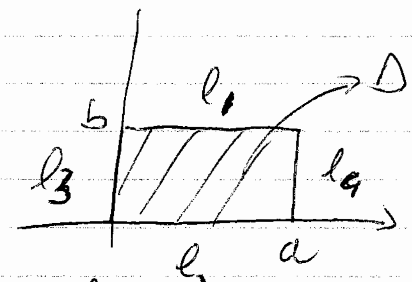
Lecture # 23

Special Domains and separation of variables

For certain domains one could solve the Laplace equation using separation of variables

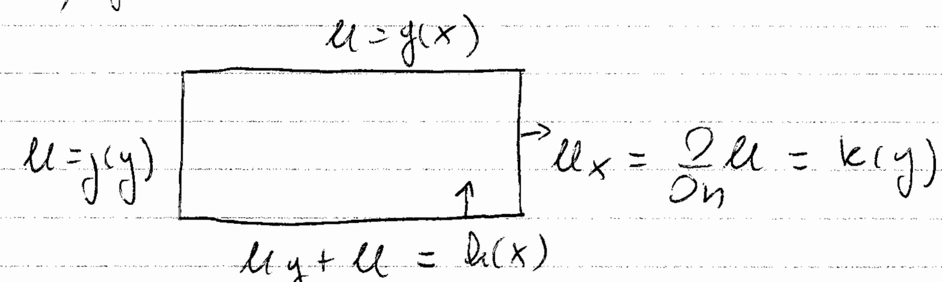
Ex 1: 2-D rectangle

$$\Delta_2 u = u_{xx} + u_{yy} = 0$$



$$D = (0,a) \times (0,b)$$

On each of the sides one can put either Dirichlet, or Neumann or Robin conditions



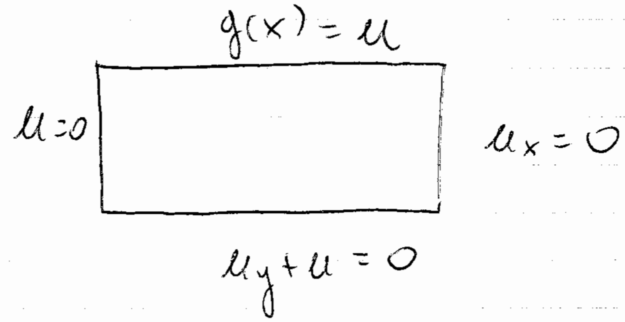
$$u_y + u = h(x)$$
$$\parallel$$
$$\frac{\partial}{\partial \vec{n}} u + u$$

u solves $\begin{cases} \Delta_2 u = 0 \\ u\big|_{\partial D} = (g, h, j, k) \end{cases} \iff u = u_1 + u_2 + u_3 + u_4$

where

$\begin{cases} \Delta u_1 = 0 \\ u_1\big|_{\ell_1} = (g,0,0,0) \end{cases}$ $\begin{cases} \Delta u_2 = 0 \\ u_2\big|_{\ell_2} = (0,h,0,0) \end{cases}$ $\begin{cases} \Delta u_3 = 0 \\ u_3\big|_{\ell_3} = (0,0,j,0) \end{cases}$ $\begin{cases} \Delta u_4 = 0 \\ u_x\big|_{\ell_4} = (0,0,0, \end{cases}$

Then the 4 problems below can be solved by separation of variables in the way discussed before.
In the book is discussed the case



$$u_1(x,y) = \sum_{n=0}^{\infty} A_n \sin \beta_n x \,(\beta_n \cosh \beta_n y - \sinh \beta_n y)$$

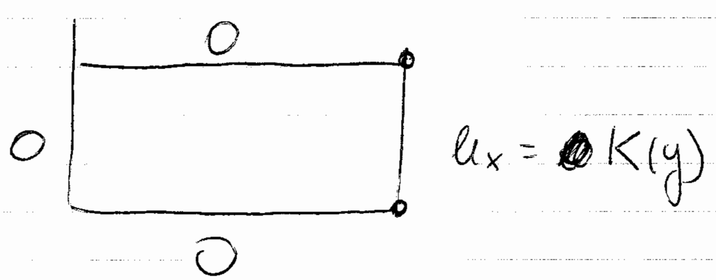
and

$$\beta_n = \left(n + \frac{1}{2}\right) \frac{\pi}{a} \qquad n = 0, 1, \dots$$

and A_n s.t.
$$g(x) = \sum_{n=0}^{\infty} A_n (\beta_n \cosh \beta_n b - \sinh \beta_n b) \sin \beta_n x$$

$$0 < x < a$$

Here let's solve u_4



We separate variables $\quad u(x,y) = X(x)\,Y(y$

$$\Delta_2 X(x)Y(y) = X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\begin{cases} \dfrac{X''}{X} = +\lambda \\ 0 \le x \le a \end{cases} \qquad \begin{cases} \dfrac{Y''}{Y} = -\lambda \\ 0 \le y \le b \end{cases}$$

In particular from the data in l_1 and l_2 it follows

$$\begin{cases} Y'' = \lambda Y \\ Y'(0) + Y(0) = 0 \\ Y(b) = 0 \end{cases}$$

this is an eigenvalue problem with Robin' Dirichlet conditions at the boundary points 0 and b

~~Assume $l_0, l_1, \dots, l_n \dots$ are the eigenvalues then using the notation in the book for RC.~~

$$\cancel{Y'(0) - a_0 Y = 0 \Rightarrow a_0 = -1}$$

$$\cancel{Y'(l) + a_l Y = 0}$$

Find eigenvalues λ_n and associated eigenfunctions $\cancel{Y_n}$ then write

$$\cancel{Y}(y) = \sum A_n \cancel{Y_n}(y)$$

Once λ_n are obtained go back to equation of X and solve

$$\begin{cases} X_n'' = +\lambda_n X_n & \text{in } (0,a) \\ X_n(0) = 0 \end{cases}$$

then put everything together

$$u_2(x,y) = \sum_{n=0}^{\infty} A_n Y_n(y) \left(B_n X_n(x) \right)$$

Finally use the fact that $u_2(a,y) = k(x)$

to go back and determine A_n and B_n.

- A similar and lengthy construction solves also $\Delta_3 u$ in D when D is a box.

<u>Dirichlet Problem in a disk</u>

$$\begin{cases} \Delta_2 u = 0 & x^2+y^2 \leq a^2 \\ u = h(\theta) & \theta \text{ in } x^2+y^2 = a^2 \end{cases}$$

We already proved that

$$\Delta_2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

We use separation of variables $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\Theta R'' + \frac{1}{r} R'\Theta + \frac{1}{r^2}\Theta'' R = 0$$

$$\Theta\left(R'' + \frac{1}{r}R'\right) + \Theta''\frac{R}{r^2} = 0$$

~~divide by~~ multiply by $\dfrac{r^2}{R\Theta}$ $\Rightarrow$

$$\frac{r^2 R'' + r R'}{R} + \frac{\Theta''}{\Theta} = 0$$

hence $\quad r^2 R'' + r R' - \lambda R = 0$

$$\Theta'' = -\lambda\Theta$$

clearly $\Theta(\theta)$ is a 2π-periodic function so we look at

$$\begin{cases} \Theta'' = -\lambda\Theta \\ \Theta(\theta) = \Theta(2\pi+\theta) \end{cases}$$

A simple calculation gives $\quad \lambda = n^2 \quad n = 1, 2 \ldots$

$$\Theta(\theta) = A\cos n\theta + B\sin n\theta$$

$$\lambda = 0 \qquad \Theta(\theta) = A$$

Go back to R

we look for $\begin{cases} r^2 R'' + rR' - n^2 R = 0 \\ R(r) = r^\alpha \end{cases}$

$$r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha = 0$$

$$r^\alpha\left(\alpha(\alpha-1) + \alpha - n^2\right) = 0$$

$$\alpha^2 - \alpha + \alpha - n^2 = 0 \qquad \alpha^2 = n^2 \qquad \alpha = \pm n$$

$$R(r) = C r^n + D r^{-n}$$

$$u(r,\theta) = \left(C r^n + D r^{-n}\right)(A\cos n\theta + B\sin n\theta)$$

$$n = 1, 2 \ldots$$

For $n=0$ we solve $r^2 R'' + r R' = 0$

one can check that constants solve it, but also $\ln r$ since $(\ln r)' = \frac{1}{r}$ $\quad (\ln r)'' = -\frac{1}{r^2}$

so we also have $u(r,\theta) = C + D\log r$

Because we do not want syngular solution at $0 = r$

Since in the disk u has to have 2 derivatives, we we discard the solutions $\ln r$ and r^{-n} so

$$(*)\quad u(r,\theta) = \underbrace{\frac{1}{2}A_0}_{\text{constant solution}} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

by the boundary condition $u(a,\theta) = h(\theta)$ we get

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

so
$$A_0 = \frac{1}{\pi}\int_0^{2\pi} h(s)\, ds$$

$$A_n = \frac{1}{a^n \pi}\int_0^{2\pi} h(s) \cos ns\, ds$$

$$B_n = \frac{1}{a^n \pi}\int_0^{2\pi} h(s) \sin ns\, ds$$

If we substitute this in $(*)$ we obtain

$$u(r,\theta) = \int_0^{2\pi} h(s)\frac{ds}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{a^n \pi} \cancel{\int_0 h(s)}$$

$$\times \int_0^{2\pi} h(s) \underbrace{(\cos ns \cos n\theta + \sin ns \sin n\theta)}_{= \cos n(\theta - s)} ds$$

$$= \int_0^{2\pi} h(s) \left\{ 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - s) \right\} \frac{ds}{2\pi}$$

Now in {.} we find

$$1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-s)}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-s)}$$

$$=1+\sum_{n=1}^{\infty} z^n \quad \text{where} \quad z=\frac{r}{a}e^{i(\theta-s)}$$

$$+\sum_{n=1}^{\infty}(\bar{z})^n \quad \text{and} \quad |z|=\frac{r}{a}<1 \quad \text{if } r<a$$

so $$=1+\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}}=$$

$$=1+\frac{z-z\bar{z}+\bar{z}-z\bar{z}}{(1-z)(1-\bar{z})}=1+\frac{z+\bar{z}-2z\bar{z}}{1-\bar{z}-z+z\bar{z}}$$

$$=1+\frac{2\operatorname{Re}z-2|z|^2}{1-2\operatorname{Re}z+|z|^2}=1+\frac{\frac{2r}{a}\cos(\theta-s)-2\frac{r^2}{a^2}}{1-2\frac{r}{a}\cos(\theta-s)+\frac{r^2}{a^2}}$$

$$=1+\frac{2ar\cos(\theta-s)-2r^2}{a^2-2ra\cos(\theta-s)+r^2}=\frac{a^2-r^2}{a^2-2ra\cos(\theta-s)+r^2}$$

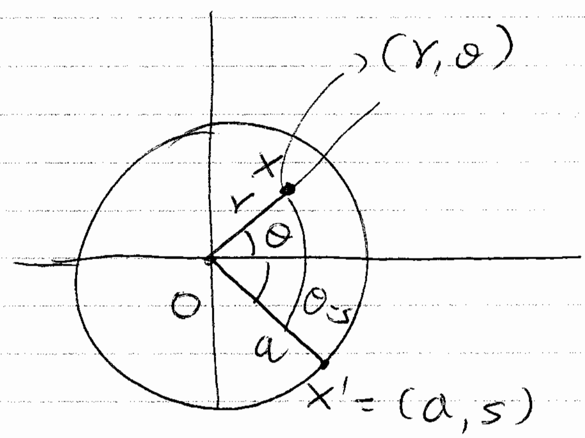
Hence

$$u(r,\theta)=(a^2-r^2)\int_0^{2\pi}\frac{h(s)}{a^2-2ra\cos(\theta-s)+r^2}\,\frac{ds}{2\pi}$$

$$\boxed{\text{Poisson Formula}}$$

<u>Remark</u>: the main point of this fact is that any harmonic function ~~continuous~~ on a disk can be written in terms of its value on the boundary of the disk!

Now let's go from polar coordinates back to ~~standard~~ Cartesian coordinates



Notice that $|x - x'|^2 = \left|(r\cos\theta, r\sin\theta) - (a\cos s, a\sin s)\right|^2$

$= \left|(r\cos\theta - a\cos s, r\sin\theta - a\sin s)\right|^2$

$= a^2 + r^2 - 2ar\cos(\theta - s)$

$$u(x) = \frac{a^2 - |x|^2}{a} \int_{|x'| = a} \frac{u(x')}{|x - x'|^2} \, d\sigma'$$

where $\sigma' = as$ = arch lenght $d\sigma' = a \, ds$

Consequences of Poisson Formula

2) Maximum principle:

$u(x)$ harmonic function in D open connected domain

b) ~~assum~~ there exists no x_0 in D s.t. $u(x_0) = \max_{\overline{D}} u$ or $u(x_0) = \min_{\overline{D}} u$, unless $u = \text{const}$.

Proof: ~~AS~~ By contradiction assume there is x_0 in D s.t.

$u(x_0) = \max_{\overline{D}} u = M$ so

$$u(x) \le u(x_0) = M \text{ for all } x \text{ in } \overline{D}$$

Because ~~D~~ D is open there exists a disk centered at x_0 and radius ε, $D_\varepsilon = D(x_0, \varepsilon)$, all contained in D.

By the Poisson formula

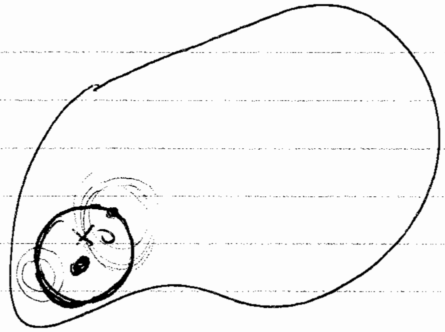
$$u(x_0) = \frac{\varepsilon^2 - |x_0|^2}{2\pi\varepsilon} \int_{|x'|=\varepsilon} \frac{u(x')}{|x_0 - x'|^2}\, d\sigma' \le M$$

$= \text{average on } D_\varepsilon$ easy

so

$$M = u(x_0) = \text{average on } D_\varepsilon \le M$$

If the average is the max then all points are max points so ~~on ∂D_ε~~ $u|_{\partial D_\varepsilon} = M$



~~By centering the circles on the pos~~

Now pick any point ~~x_0~~ X inside D_ε.
Then by taking a smaller radius
we can find a new disk ~~centered at~~
~~x_0~~ $D(x_0, \tilde{\varepsilon})$, $\tilde{\varepsilon} < \varepsilon$ s.t. X is on the boundary
of $D(x, \tilde{\varepsilon})$. By the above argument $u(x) = u(x_0) = M$
so $u = M$ on D_ε. By covering D with such disks we
~~conclude~~ finally get $u \equiv M$ in D.

(2) Mean Value problem

Assume $X = 0$ then the Poisson Formula reads

$$u(0) = \frac{\cancel{a^2}}{2\pi a} \int_{|x'|=a} \frac{u(x')}{\cancel{a^2}} \, d\sigma' =$$ Average of u on circle of radius a.

(3) Differentiability: If u is an harmonic function in any open set D of the plane, then u has all the partial derivatives in any order.

Proof: Use Poisson formula:

Let X ~~be~~ in D an $D_\varepsilon = D(x, \varepsilon)$ included in D

~~u(x)=~~ a^2 then

$$u(x) = \frac{\varepsilon^2 - |x|^2}{2\pi\varepsilon} \int_{|x'|=\varepsilon} \frac{u(x')}{|x-x'|^2}\, d\sigma'$$

Because the function we are integrating

$f(x,x') = \dfrac{u(x')}{|x-x'|^2}$ is ~~continuous~~ ~~in~~ non singular on $|x'| = \varepsilon$ and continuous ~~in~~ both variables

then one can pass any derivative inside since $\frac{1}{|x-x'|^2}$ holds all of them.