MATH 152, FALL 2004: FINAL

There are five problems. Do all of them. Total score: 160 points.

Problem #1, *(25 points)*
For both of the following functions f on $[0, l]$, state whether the Fourier cosine series on $[0, l]$ converges in each of the following senses: uniformly, pointwise, in L^2. If the Fourier series converges pointwise, state what it converges to for each $x \in [0, l]$. Make sure that you give the reasoning that led you to the conclusions.

1. $f(x) = x(\sin(\pi x/l))^2$,
2. $f(x) = 0$, for $0 \leq x \leq l/2$, and $f(x) = 1$ for $l/2 < x \leq l$.

Problem #2, *(25 points total)*

1. *(8 points)* Find the general solution of the PDE
$$u_x + 2yu_y = 0$$
on $\mathbb{R}_x \times \mathbb{R}_y$.
2. *(7 points)* Now impose in addition that $u(0, y) = y$. Find u explicitly.
3. *(10 points)* Consider the PDE
$$u_x + 2yu_y = x$$
on $\mathbb{R}_x \times \mathbb{R}_y$. Find its general solution.

Problem #3, *(35 points total)*
Consider the differential operator
$$A = -\frac{d}{dx}(x^2 \frac{d}{dx})$$
on twice differentiable functions f on $[0, l]$ which satisfy Dirichlet boundary conditions $f(0) = f(l) = 0$. That is, for these functions f, $Af = -(x^2 f'(x))'$. Let $(f, g) = \int_0^l f(x)\overline{g(x)}\, dx$ denote the standard inner product on functions.

1. *(10 points)* Show that A is positive: $(Af, f) \geq 0$ for all f satisfying the conditions above.
2. *(5 points)* Use 1. to prove that all the eigenvalues of A are real and positivek.
3. *(10 points)* Show that A is symmetric: $(Af, g) = (f, Ag)$ for all functions f, g satisfying the boundary conditions.
4. *(10 points)* Use 3. to prove that all eigenvectors associated to different eigenvalues are orthogonal.

Problem #4, *(35 points total)*
Consider the heat equation $u_t = ku_{\theta\theta}$ in a thin ring, and suppose that the initial temperature of the ring is $u(\theta, 0) = \phi(\theta)$, $\theta \in [0, 2\pi]$, ϕ is a given function. Since physically θ is defined up to the addition of integer multiples of 2π, we may consider u as a 2π-periodic function of θ.

1. *(9 points)* The total heat energy in the ring at time t is
$$Q(t) = \int_0^{2\pi} u(\theta, t)\, d\theta.$$
Using the PDE, show that $Q(t)$ is a constant (independent of t). (Hint: consider dQ/dt.)
2. *(12 points)* Using the separation of variables, show that the general solution of the heat equation is
$$A_0/2 + \sum_{n=1}^{\infty} e^{-n^2 kt}(A_n \cos(n\theta) + B_n \sin(n\theta)),$$
and find A_n, B_n in terms of ϕ.
3. *(7 points)* Find Q in terms of the A_n and B_n. What does the conservation of heat energy, as in 1., correspond to in the series solution 2.?
4. *(7 points)* Find $\lim_{t\to+\infty} u(\theta, t)$ in terms of ϕ. Interpret the result physically.

Problem #5, *(40 points total)*
In this problem we compare Laplace's equation and the wave equation in a half-space. For the sake of uniformity of notation, we let (x, y) be our variables in $\mathbb{R}^2$, work in the half plane $y \geq 0$, and consider
$$(L)\ u_{xx} + u_{yy} = 0, \text{ respectively } (W)\ u_{xx} - u_{yy} = 0,$$
with
$$u(x, 0) = \phi(x),\ u_y(x, 0) = \psi(x),$$
where ϕ, ψ are given test functions. Thus, in the wave equation, you can think of y as the time variable, and we have the usual two initial conditions, while in Laplace's equation this amount to imposing *both* Dirichlet and Neumann boundary conditions at $y = 0$. We assume that u is a tempered distribution.

1. *(10 points)* Take the Fourier transform in x of both PDE's and the conditions at $y = 0$. That is, let $\hat{u}(\xi, y) = \mathcal{F}_x u(\xi, y) = \int_{\mathbb{R}} e^{-ix\xi} u(x, y)\, dx$. Recall that $\mathcal{F}_x(D_x u) = \xi \mathcal{F}_x u$, where $D_x = \frac{1}{i}\frac{\partial}{\partial x}$, and show that the PDE's become
$$(\hat{L})\ -\xi^2 \hat{u} + \hat{u}_{yy} = 0, \text{ respectively } (\hat{W})\ -\xi^2 \hat{u} - \hat{u}_{yy} = 0.$$
What are the conditions at $y = 0$?
2. *(12 points)* Solve the ODE's $(\hat{L})$, resp. $(\hat{W})$, together with the conditions at $y = 0$.
3. *(8 points)* Show that for $(\hat{W})$, the solution is a tempered distribution in ξ and y. (Recall that every continuous function v such that $|v(\xi, y)| \leq C(1 + |\xi|^2 + |y|^2)^N$ for some C and N is a tempered distribution.) By taking the inverse Fourier transform, write u as a sum of two terms, each of which is the convolution of a source function (which you do not have to calculate explicitly) and ϕ, resp. ψ.
4. *(10 points)* Show that for $(\hat{L})$, in general, the solution is exponentially increasing in $y \geq 0$, hence is not a tempered distribution. What equation must ϕ and ψ satisfy to make the exponentially increasing term in $\hat{u}$ disappear? In particular, show that ϕ determines ψ if $\hat{u}$ is polynomially bounded, i.e. it suffices to specify the Dirichlet boundary condition ϕ. Again, in this situation, write u as a convolution.

Solution for final

1) $f : [0,\ell] \to \mathbb{R}$. Cosine F.S. on $[0,\ell]$

1) $f(x) = x\left(\sin\left(\frac{\pi x}{\ell}\right)\right)^2$

a) Uniform – **YES**

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Conditions for unif. conv. of classic F.S. are f, f' exist, are continuous and the B.C. are satisfied. We have $f(0) = f(\ell) = 0$.

Our e.funs $\cos\left(\frac{\pi n x}{\ell}\right) = \begin{cases} 1, & x = 0 \\ -1, & x = \ell, \end{cases}$ $n > 0$. So $\sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{\ell}\right) = \sum_{n=1}^{\infty} A_n$ at $x = 0$, $\sum_{n=1}^{\infty} (-1)^n A_n$ at $x = \ell$, so B.C. can be satisfied w/ appropriate choice of A_0.

b) Pointwise – YES

Condition for pointwise conv. of classic F.S. is that both f and f' are piecewise continuous.

Clearly f cont.

$f' = \left(\sin\left(\frac{\pi x}{\ell}\right)\right)^2 + x \cdot 2\sin\left(\frac{\pi x}{\ell}\right) \cdot \frac{\pi}{\ell}\cos\left(\frac{\pi x}{\ell}\right)$.

also continuous. $\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n x}{\ell} = f(x)$.

c) L^2 – YES

Cond. for L^2 conv. is that $f \in L^2([0,\ell])$.

$$\int_0^{\ell} \left(x\left(\sin\left(\tfrac{\pi x}{\ell}\right)\right)^2\right)^2 dx = \int_0^{\ell} x^2 \sin^4\left(\tfrac{\pi x}{\ell}\right) dx \le \int_0^{\ell} x^2\, dx = \tfrac{1}{3}\ell^3 < \infty.$$

2) $f(x) = \begin{cases} 0, & 0 \le x \le \ell/2 \\ 1, & \frac{\ell}{2} < x \le \ell \end{cases}$

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a) Uniform – **NO**. The conditions given above for unif. convergence are not necessary!

~~F is not even continuous, let alone its derivative.~~ ~~But in a sense you are right because~~ the series is continuous and uniform limit of cont. functions is continuous.

b) Pointwise – YES.

This is the reason!

f is continuous everywhere but $\frac{\ell}{2}$. $f' = 0$ everywhere but $\frac{\ell}{2}$. So both piecewise cont.

~~Pointwise convergence~~ at $x = \frac{\ell}{2}$!

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n x}{\ell} = \begin{cases} 0, & 0 \le x < \ell/2 \\ 1/2, & x = \ell/2 \\ 1, & \ell/2 < x \le \ell \end{cases}$$

cont. $\Rightarrow$

(#1 cont.)

c) L^2 - YES

✓ Again, just need $f \in L^2([0,\ell])$

$$\int_0^\ell f^2\,dx = \int_{\ell/2}^{\ell} 1^2\,dx = \frac{\ell}{2} < \infty.$$

(2) 1) $u_x + 2y u_y = 0$

Use coord. method. Const. lines are $\frac{dy}{dx} = \frac{2y}{1}$.

$\Rightarrow \frac{dy}{y} = 2dx$

$\Rightarrow \log y = 2x + c \Rightarrow y = Ce^{2x}$

(8) So u is constant along $e^{-2x}y$

$$\boxed{\Rightarrow u(x,y) = f(e^{-2x}y)}$$

Test: $u_x = f' \cdot -2e^{-2x}y$, $u_y = f' \cdot e^{-2x}$

$u_x + 2y u_y = -2e^{-2x}y f' + 2y e^{-2x} f' = 0$

2) $u(0,y) = y$

(7) $u(0,y) = f(y) = y$

$$\boxed{\Rightarrow u(x,y) = z\big|_{e^{-2x}y} = e^{-2x}y}$$

3) $u_x + 2y u_y = x$

Now instead of being constant along $y = Ce^{2x}$, we have the derivs in those directions equal to x.

(10) Do a coord. transform, $x' = x$

$y' = ye^{-2x}$

$u_x = \frac{\partial}{\partial x'} u \cdot \frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'} u \cdot \frac{\partial y'}{\partial x} = u_{x'} \cdot 1 + u_{y'} \cdot -2ye^{-2x}$

$u_y = \frac{\partial}{\partial x'} u \cdot \frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'} u \cdot \frac{\partial y'}{\partial y} = u_{x'} \cdot 0 + u_{y'} \cdot e^{-2x}$

So $u_x + 2y u_y = u_{x'} \cancel{-2ye^{-2x} u_{y'}} + \cancel{2ye^{-2x} u_{y'}} = x'$ $\Rightarrow$ cont.

(2.3 cont.)

$$u_{x'} = x'$$

$$\Rightarrow u(x',y') = \frac{1}{2}x'^2 + f(y')$$

$$\boxed{\Rightarrow u(x,y) = \frac{1}{2}x^2 + f(ye^{-2x})}$$

(3) $A = -\frac{d}{dx}\left(x^2 \frac{d}{dx}\right)$, f twice diff., $f(0) = f(\ell) = 0$

1) Show A positive

$$(Af, f) = \int_0^\ell Af \cdot \bar{f}\, dx = \int_0^\ell -(x^2 f'(x))' \bar{f}\, dx$$

$$= \left[\bar{f} \cdot -x^2 f'(x)\right]_0^\ell + \int_0^\ell x^2 f'(x) \cdot \bar{f}'\, dx$$

(10)

$$= (\cancel{\bar{f}(\ell) \cdot -\ell^2 f'(\ell)} - \cancel{\bar{f}(0) \cdot 0^2 \cdot f'(0)}) + \int_0^\ell x^2 |f'(x)|^2 dx$$

$0,\ \bar{f}(\ell) = 0$ $\quad 0$

$$= \int_0^\ell |x f'(x)|^2 dx.$$

$$|x f'(x)|^2 \geq 0 \quad \forall x \Rightarrow \int_0^\ell |x f'(x)|^2 dx \geq 0.$$

$$\Rightarrow (Af, f) \geq 0.$$

2) λ is an e. value of A if $Af = \lambda f$ for some f

Let f be an e. fcn.

$$(Af, f) = (\lambda f, f) = \lambda (f, f)$$

(5)

$$(Af, f) \geq 0 \Rightarrow \lambda (f, f) \geq 0$$

$\lambda = 0$?

$$(f, f) \geq 0 \Rightarrow \lambda \geq 0.$$

(λ cannot be complex because $(f,f) \in \mathbb{R}$ and we showed $(Af, f) \in \mathbb{R}$

3) Show $(Af, g) = (f, Ag)$ $\forall f, g$ satisfying B.C.

$$(Af, g) = \int_0^\ell Af \cdot \bar{g}\, dx = \int_0^\ell -(x^2 f'(x))' \cdot \bar{g}\, dx$$

$$= \left[\bar{g} \cdot -x^2 f'(x)\right]_0^\ell - \int_0^\ell -x^2 f'(x) \cdot \bar{g}'\, dx$$

(10)

$$= \left[\cancel{\bar{g}(\ell) \cdot -\ell^2 f'(\ell)} - \cancel{\bar{g}(0) \cdot 0^2 \cdot f'(0)}\right] + \int_0^\ell x^2 f'(x) \bar{g}'(x)\, dx$$

$0,\ \bar{g}(\ell) = 0$ $\quad 0$

$$(f, Ag) = \int_0^\ell f \cdot (\overline{Ag})\, dx = \int_0^\ell f \cdot \overline{(-x^2 g'(x))'}\, dx$$

$\Rightarrow$ cont.

(3.3 cont.)

$$= \int_0^\ell f \cdot \left(-x^2 \bar{g}'(x)\right)' dx$$

$$= \left[f\ell \cdot -x^2 \bar{g}'\right]_0^\ell - \int_0^\ell f' \cdot -x^2 \bar{g}'(x)\, dx$$

$$= \left[\cancel{f(\ell) \cdot -\ell^2 \bar{g}'(\ell)}_{0,\ f(\ell)=0} - \cancel{f(0)\, 0^2\, \bar{g}'(0)}_{0}\right] + \int_0^\ell x^2 f'(x)\, \bar{g}'(x)\, dx.$$

Note, these two results are ident. so $(Af, g) = (f, Ag)$.

4) Prove all e. vectors for diff e. values are orthog.

<u>Pf</u> Let λ_1, λ_2 be e. values, $\lambda_1 \neq \lambda_2$.

Let f_1, f_2 be any e. fcns assoc. w/ λ_1, λ_2 respectively.

Then $Af_1 = \lambda_1 f_1$

$Af_2 = \lambda_2 f_2$

We want to show $(f_1, f_2) = 0$.

$$(f_1, f_2) = \left(\frac{\lambda_1 f_1}{\lambda_1}, f_2\right)$$

$$= \left(\frac{Af_1}{\lambda_1}, f_2\right)$$

$$= \left(\frac{f_1}{\lambda_1}, Af_2\right) \quad \text{(from part 3)}$$

$$= \left(\frac{f_1}{\lambda_1}, \lambda_2 f_2\right)$$

(3.4 cont.)

So this says that

$$\int_0^\ell f_1 \overline{f_2}\, dx = \int_0^\ell \frac{f_1}{\lambda_1} \cdot (\overline{\lambda_2 f_2})\, dx$$

$$= \frac{\lambda_2}{\lambda_1} \int f_1 \overline{f_2}\, dx$$ (showed in part 2 that λ's are real)

$$\Rightarrow \left(\frac{\lambda_2}{\lambda_1} - 1\right) \int_0^\ell f_1 \overline{f_2}\, dx = 0.$$

(10)

This is true only if $\lambda_1 = \lambda_2$ (impossible, we just said $\lambda_1 \neq \lambda_2$)

or if $\int_0^\ell f_1 \overline{f_2}\, dx = 0$.

This is equiv. to saying $(f_1, f_2) = 0$

$\Rightarrow f_1, f_2$ orthog.

④ $u_t = k u_{\theta\theta}$ in a thin ring

$u(\theta, 0) = \phi(\theta), \quad \theta \in [0, 2\pi]$

$u(\theta + 2\pi, t) = u(\theta, t)$

1) $Q(t) = \text{heat energy} = \int_0^{2\pi} u(\theta, t)\, d\theta$

We note that because u is 2π periodic, u_θ is also 2π periodic. $\left(u_\theta(\theta + 2\pi, t) = \lim_{\delta\theta \to 0} \frac{u(\theta + 2\pi + \delta\theta, t) - u(\theta + 2\pi, t)}{\delta\theta} = \lim_{\delta\theta \to 0} \frac{u(\theta + \delta\theta, t) - u(\theta, t)}{\delta\theta} = u_\theta(\theta, t) \right)$

Then $\frac{dQ}{dt} = \frac{d}{dt} \int_0^{2\pi} u(\theta, t)\, d\theta = \int_0^{2\pi} \frac{d}{dt} u(\theta, t)\, d\theta$ (u is nice, both partials exist, so this is OK)

$= \int_0^{2\pi} u_t(\theta, t)\, d\theta = \int_0^{2\pi} k u_{\theta\theta}(\theta, t)\, d\theta$ (PDE)

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$= k [u_\theta]_0^{2\pi} = k(u_\theta(2\pi, t) - u_\theta(0, t))$

$= 0.$ (because u_θ 2π periodic) ✓

Thus Q is constant over time.

2) Let $u(\theta, t) = \Theta(\theta) \cdot T(t)$.

Then $\Theta T' = k \cdot \Theta'' \cdot T \Rightarrow \frac{T'}{kT} = \frac{\Theta''}{\Theta} = -\lambda$

$\Rightarrow \Theta'' = -\lambda \Theta$.

For $\lambda > 0$, $\lambda = \beta^2 \Rightarrow \Theta'' = -\beta^2 \Theta \Rightarrow \Theta(\theta) = C \cos \beta\theta + D \sin \beta\theta$

Need $\Theta(0) = \Theta(2\pi) \Rightarrow \Theta(0) = C$

$\Theta(2\pi) = C \cos 2\pi\beta + D \sin 2\pi\beta$ ✓

$\Rightarrow \beta = n$

(Note, condition is actually $\Theta(\theta) = \Theta(\theta + 2\pi)$, after crawling through algebra, this reduces to above condition – see attached scrap work)

→ cont.

(4.2 cont.)

So $\Theta_n(\theta) = C_n \cos n\theta + D_n \sin n\theta$

$\lambda = +n^2$

$T' = -\lambda k T \Rightarrow T(t) = Ae^{-\lambda k t} = Ae^{-n^2 k t}$

$\lambda = 0$, $\Theta'' = 0 \Rightarrow \Theta(\theta) = Cx + D$ on $(0, 2\pi)$

$\Theta(0) = \Theta(2\pi) \Rightarrow \Theta(0) = D$, $\Theta(2\pi) = 2\pi C + D$

$\Rightarrow C = 0$

$\Theta_0(\theta) = D_0$

$T' = 0 \Rightarrow T_0(t) = A_0$

$\lambda < 0$ will get us cosh and sinh which will not satisfy B.C.

So Then $u(\theta, t) = A_0' \cdot D_0 + \sum_{n=1}^{\infty} e^{-n^2 k t} (C_n \cos n\theta + D_n \sin n\theta)$

$= \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 k t} (A_n \cos n\theta + B_n \sin n\theta)$

(12)

$u(\theta, 0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) = \phi(\theta)$

Clearly, this is a standard F.S.

So $A_0 = \frac{2}{2\pi} \int_0^{2\pi} \phi(\theta)\,d\theta = \frac{1}{\pi} \int_0^{2\pi} \phi(\theta)\,d\theta$

$A_n = \frac{1}{\pi} \int_0^{2\pi} \cos n\theta \cdot \phi(\theta)\,d\theta$

$B_n = \frac{1}{\pi} \int_0^{2\pi} \sin n\theta\, \phi(\theta)\,d\theta$

3) $Q(t) = \int_0^{2\pi} u(\theta, t)\,d\theta = \int_0^{2\pi} \left(\frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 k t} (A_n \cos n\theta + B_n \sin n\theta) \right) d\theta$

$= \frac{A_0}{2} \cdot 2\pi + \sum_{n=1}^{\infty} e^{-n^2 k t} \int_0^{2\pi} (A_n \cos n\theta + B_n \sin n\theta)\,d\theta$

$= A_0 \pi + \sum_{n=1}^{\infty} e^{-n^2 k t} \left[\cancel{\frac{A_n \sin n\theta}{n} - \frac{B_n \cos n\theta}{n}} \right]_0^{2\pi}$ 0, 2π periodic

(OK to switch Σ and $\int$ because all fcns are very nice)

$= A_0 \pi$

$\Rightarrow$ (cont.)

(4.3 cont.)

So the fact that heat is conserved falls directly out from the fact that the solution ~~e~~ admits a series form. The way to view it is that each of the cos/sin terms have exactly zero net heat energy. So these terms can be multiplied by any function of ~~t~~. → very good!

(7)

(~~constant in terms of θ~~)

The key is that the $\lambda = 0$ term (which does have heat energy) ~~a~~ has a constant t part. This causes the overall heat energy to be indep. of t.

4) $$\lim_{t\to\infty} u(\theta,t) = \lim_{t\to\infty}\left(\frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-n^2kt}(A_n\cos n\theta + B_n\sin n\theta)\right)$$

$$= \frac{A_0}{2} + \lim_{t\to\infty}\sum_{n=1}^{\infty} e^{-n^2kt}(A_n\cos n\theta + B_n\sin n\theta)$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty}\lim_{t\to\infty} e^{-n^2kt}(A_n\cos n\theta + B_n\sin n\theta)$$

(can swap limits because we have unif. conv. - u, u' both contin.)

(7) $$= \frac{A_0}{2} = \frac{1}{2\pi}\int_0^{2\pi}\phi(\theta)\,d\theta.$$

good!

Note, this is a constant value equal to the average value of ϕ. So the heat equation says that at steady state, the initial heat energy all stays within the ring, and it gets evenly distributed in the ring.

(5) (L) $U_{xx} + U_{yy} = 0$ $\quad$ (W) $U_{xx} - U_{yy} = 0$ $\quad \Big\}$ $u(x,0) = \Phi(x)$, $u_y(x,0) = \Psi(x)$

1) $F_x\{U_{xx} \pm U_{yy}\} = -\xi^2 \hat{u}(\xi,y) \pm \frac{d^2}{dy^2}\hat{u}(\xi,y) = 0$

$F_x\{u(x,0)\} = \hat{u}(\xi,0) = \hat{\Phi}(\xi)$

$F_x\{u_y(x,0)\} = \frac{d}{dy}\hat{u}(\xi,0) = \hat{\Psi}(\xi)$

(10)

$$\Rightarrow (\hat{L}) \quad -\xi^2\hat{u} + \hat{u}_{yy} = 0 \qquad \hat{u}(\xi,0) = \hat{\Phi}(\xi)$$
$$(\hat{W}) \quad -\xi^2\hat{u} - \hat{u}_{yy} = 0 \qquad \hat{u}_y(\xi,0) = \hat{\Psi}(\xi)$$

2) $(\hat{L})$ $\hat{u}_{yy} = \xi^2\hat{u} \Rightarrow \hat{u}(\xi,y) = A(\xi)e^{-\xi y} + B(\xi)e^{\xi y}$

$\hat{u}(\xi,0) = A(\xi) + B(\xi) = \hat{\Phi}(\xi) \Rightarrow A = \hat{\Phi} - B$

$\hat{u}_y(\xi,y) = (-A(\xi)\cdot\xi e^{-\xi y} + B(\xi)\xi e^{\xi y})$

$\hat{u}_y(\xi,0) = -A(\xi)\cdot\xi + B(\xi)\cdot\xi = \hat{\Psi}(\xi)$

$\Rightarrow -(\hat{\Phi}-B)\xi + B\xi = -\xi\hat{\Phi} + 2B\xi = \hat{\Psi} \Rightarrow B(\xi) = \frac{\hat{\Psi} + \xi\hat{\Phi}}{2\xi} = \frac{\hat{\Psi}}{2\xi} + \frac{1}{2}\hat{\Phi}$

Then $A(\xi) = \hat{\Phi}(\xi) - B(\xi)$

$= \frac{1}{2}\hat{\Phi}(\xi) - \frac{\hat{\Psi}(\xi)}{2\xi}$ ✓

So then $\hat{u}(\xi,y) = \frac{1}{2}\left(\hat{\Phi}(\xi) - \frac{\hat{\Psi}(\xi)}{\xi}\right)e^{-\xi y} + \frac{1}{2}\left(\hat{\Phi}(\xi) + \frac{\hat{\Psi}(\xi)}{\xi}\right)e^{\xi y}$

$(\hat{W})$ $\hat{u}_{yy} = -\xi^2\hat{u} \Rightarrow \hat{u}(\xi,y) = A(\xi)\cos\xi y + B(\xi)\sin\xi y$

$\hat{u}(\xi,0) = A(\xi) = \hat{\Phi}(\xi)$

(12) $\hat{u}_y(\xi,y) = -\xi A(\xi)\sin\xi y + \xi B(\xi)\cos\xi y$

$\hat{u}_y(\xi,0) = \xi B(\xi) = \hat{\Psi}(\xi)$

So $\hat{u}(\xi,y) = \hat{\Phi}(\xi)\cos\xi y + \frac{1}{\xi}\hat{\Psi}(\xi)\sin\xi y$

(#5 cont.)

3) Because ϕ, ψ are test functions, ϕ, ψ are also in $\mathcal{S}$ (Schwartz fcns). F.T. maps $\mathcal{S} \to \mathcal{S}$ so $\hat\phi, \hat\psi$ are also in $\mathcal{S}$. $\phi, \psi \in C^\infty$ (test fcn) so $\hat\phi, \hat\psi$ also are nice.

So $|\hat{U}| = |\hat\phi(\xi) \cos \xi y + \frac{1}{\xi} \hat\psi(\xi) \sin \xi y|$ is continuous

($\lim_{\xi \to 0} \frac{1}{\xi} \sin \xi y = \lim_{\xi \to 0} \frac{y \cos \xi y}{1} = y$ is well defined) good!

This needs a bit more care but you got the main point

~~For any finite ball $B(0,r)$, easy to choose C and N.~~

Choose $N = 1$ for simplicity. ($\hat{U}$ is bounded because it's continuous)

$\lim_{\xi \to \infty} \hat{U} = 0$ because $\hat\phi, \& \hat\psi$ both Schwartz fcns.

$\lim_{y \to \infty} \hat{U}$ does not exist, but $|\hat{U}(\xi, y)| \leq |\hat\phi(\xi) + \frac{1}{\xi} \hat\psi(\xi)|$ away from $\xi = 0$.

So choose $C = \max_{B(0,r) \setminus 0} \frac{|\hat\phi + 1/\xi\, \hat\psi(\xi)|}{1 + |\xi|^2 + |y^2|}$ and $\hat{U}$ is tempered.

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$$u(x,y) = \mathcal{F}^{-1}\{\hat{U}\} = \mathcal{F}^{-1}\{\hat\phi(\xi) \cos \xi y + \tfrac{1}{\xi} \hat\psi \sin \xi y\}$$
$$= (\phi * \mathcal{F}^{-1}\{\cos \xi y\})(x,y)$$
$$+ (\psi * \mathcal{F}^{-1}\{\tfrac{1}{\xi} \sin \xi y\})(x,y)$$

Of course, $\mathcal{F}^{-1}\{\cos \xi y\} = \frac{1}{2}\delta(x-y) + \frac{1}{2}\delta(x+y)$

other term is $\frac{1}{2c} H(x^2 - c^2 t^2)$ (standard wave equ soln)

4) Clearly $\hat{U}(\xi, y)$ for $\hat{2}$ is not bounded in terms of y in general.

Imagine $\hat\phi(\xi) = 1$, $\hat\psi(\xi) = 0$. Then

$$\hat{U}(\xi, y) = \tfrac{1}{2} e^{-\xi y} + \tfrac{1}{2} e^{\xi y} \xrightarrow{y \to \infty} \infty$$

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So we must have $\hat\phi(\xi) = -\frac{\hat\psi(\xi)}{\xi}$ to have the $e^{\xi y}$ term disappear.

So if $\hat{U}$ is polynomially bounded $\Rightarrow \hat\phi(\xi) = -\frac{\hat\psi(\xi)}{\xi}$ (need positive exponential gone) $\Rightarrow$ cont.

$\Rightarrow \xi \hat\phi(\xi) = -\hat\psi(\xi) \Rightarrow$ So ψ is determined by ϕ

(5.4 cont.)

Then $\hat{U}(\xi, y) = \hat{\Phi}(\xi)\, e^{-\xi y}$

So $u(x,y) = \left(\Phi * \mathcal{F}^{-1}\{e^{-\xi y}\}\right)(x,y)$

(possibly need to work in terms of $e^{-|\xi| y}$ to get this to work properly).