Lecture #18

We recall briefly that if

$$L^2(\mathbb{R}) = \{ f : [a,b] \longrightarrow \mathbb{R} \,/\, \int_a^b f^2(x)\,dx < \infty \}$$

then f, g in $L^2(\mathbb{R})$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\,dx$$

If

$$L^2(\mathbb{C}) = \{ f : [a,b] \longrightarrow \mathbb{C} \,/\, \int_a^b f(x) \bar{f}(x)\,dx < \infty \}$$

then for f, g in $L^2(\mathbb{C})$

$$\langle f, g \rangle = \int_a^b f(x)\, \bar{g}(x)\,dx$$

Definition: In both cases we say that

f orthogonal to g iff $\langle f, g \rangle = 0$

Question: How do we find ~~an~~ orthogonal functions?

Consider the operator $A = -\frac{d^2}{dx}$ and let $X_1(x)$ and $X_2(x)$ two real valued eigenfunctions

$$-X_i'' = \lambda_i X_i \qquad i = 1, 2 \quad , \quad \lambda_1 \neq \lambda_2$$

The following identity is true

(∀) $-X_1''X_2 + X_1X_2'' = (-X_1'X_2 + X_1X_2')'$. ②

to check this one just writes explicitely the RHS by taking derivatives. Using (∀) one gets

$$\int_a^b (-X_1''X_2 + X_1X_2'')\,dx = \underbrace{(-X_1'X_2 + X_1X_2')\Big|_a^b}_{\text{boundary conditions}}$$

$\boxed{\text{Green's Second identity}}$

Assum B.C. are such that RHS $= 0$, then

$$LHS = \int_a^b (\lambda_1 X_1X_2 - \lambda_2 X_1X_2)\,dx$$

$$= \underset{\neq 0}{(\lambda_1-\lambda_2)} \int_a^b X_1X_2\,dx = (\lambda_1-\lambda_2)(X_1,X_2)$$

orthogonality $X_1 \perp X_2$.

BC:

Case 1 Dirichlet $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$

Case 2: Neumann : $X_{1,x}(a) = X_{1,x}(b) = X_{2,x}(a) = X_{2,x}(b)$.

Case 3) Periodic $X_i(a) = X_i(b)$

$X_{ix}(d) = X_{ix}(b)$

Case 4) Robin Conditions $\Longrightarrow$ also zero

$u_x(a) - a_0u(a) = 0$ $X'(a) = a_0X(a)$

$u_x(b) + a_1u(e) = 0$ $X'(b) = a_1X(b)$ Q.E.D.

$X_i'(a) = a_0X_i(a)$

$-X_i(a) = \frac{X_i'(a)}{a_0}$ $X_1'X_2(b) = X_2'X_1(b)$

So in all 4 cases one has orthogonality.

<u>Remark</u>: the RHS is not always zero, one can come up with all sorts of examples, for instance:

$X(a) = 2X(b)$ Then the condition becomes
$X'(a) = X'(b)$

$$-X_1'(a)X_2(a) + X_1(a)X_2'(a) + X_1'(b)X_2(b) + X_1(b)X_2'($$

$$= -X_1'(a)X_2(a) + X_1(a)X_2'(a) + X_1'(a)\frac{1}{2}X_2(a) - \frac{1}{2}X_1(a)X_2'(a$$

$$= -\frac{1}{2}X_1'(a)X_2(a) + \frac{1}{2}X_1(a)X_2'(a)$$ which in general is not zero.

To obtain orthogonality for eigenfunctions one needs:

Assume the pair of general boundary conditions

$$(GBC)\quad \begin{cases} \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0 \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0 \end{cases}$$

<u>Definition</u>: We say that (GBC) are <u>symmetric</u> if for any two functions f, g with (GBC) one has

$$f'(x)g(x) - f(x)g'(x)\Big|_a^b = 0$$

Remark 1: Dirichlet, Neumann, Periodic and Robin are symmetric boundary conditions.

Remark 2: a) All eigenfunctions with symmetric BC that have different eigenvalues are orthogonal.

b) if a function f is written as a linear combination of eigenfunctions as ~~above~~ (in a), then the coefficients are completely determined.

To see this assume

$$f(x)=\sum_{n=1}^{k} a_n X_n(x)$$

where $X_n(x)$ are eigenfunctions s.t. $\lambda_n \neq \lambda_m$ $n \neq m$ and symmetric BC.

$$\langle f(x), X_m(x)\rangle = \sum_{n=1}^{k} a_n \langle X_n, X_m\rangle$$

$$= a_m \langle X_m, X_m\rangle$$

$$a_m = \frac{\int_a^b f(x) X_m(x)\,dx}{\int_a^b X_m^2(x)\,dx}$$

We say that

$\{X_1, \ldots, X_n\}$ is orthogonal if $(X_n, X_m)=0$ if $m \neq n$

It is orthonormal if $(X_m, X_m) = 1$.

The complex case

Theorem: If one has a symmetric BC, all the eigenvalues of

$$-\frac{d^2}{dx^2}$$ are real valued.

Moreover ~~are~~ the set of complex linear combination of the eigenfunctions are the same as The set of complex linear combination of the real eigenfunctions.

Proof: Assume λ is eigenvalue and X eigenfunction. Then

$$-X'' = \lambda X$$ and complex conjugate

$$-\overline{X''} = \overline{\lambda}\,\overline{X}$$

$$\Longleftrightarrow \quad -\overset{\|}{\overline{X}}{}'' = \overline{\lambda}\,\overline{X} \quad \Longrightarrow \overline{\lambda} \text{ is also eigenvalue}$$

As above ~~where we~~ we have

$$\int_a^b \left(-X''\overline{X} + \overline{X}''X\right)dx = \left(-X'\overline{X} + X\overline{X}'\right)\Big|_a^b$$

if we have symmetric BC $\Longrightarrow$

$$(\lambda-\overline{\lambda})\int_a^b X\overline{X}\,dx = 0$$

$$\|$$

$$(\lambda-\overline{\lambda})\int_a^b |X|^2\,dx$$

because $X \not\equiv 0$ necessarily $\lambda - \bar{\lambda} = 0$

$$\lambda - \bar{\lambda} = \cancel{\alpha} + i\beta - \cancel{\alpha} - i\beta = 0$$

$\Rightarrow \beta = 0 \quad \Rightarrow \quad \lambda$ is real!

Now let $X_n = Y_n + i Z_n$ be eigenfunctions

$$\sum_{n=1}^{k} c_n X_n =$$ for c_n complex

$$= \sum_{n=1}^{k} c_n Y_n + i c_n Z_n$$ ~~Y_n, Z_n are~~

where Y_n and Z_n are real eigenfunctions since

$$-X_n'' = \lambda X_n \iff -Y_n'' - i Z_n'' = \lambda (Y_n + i Z_n$$

because λ is real it follows

$$-Y_n'' = \lambda Y_n \text{ and } -Z_n'' = \lambda Z_n \text{ !}$$

Q.E.D.

<u>Theorem 2</u>: Assume symmetric BC.

Then if $(\forall f)$ $f(x) f'(x) \Big|_{x=a}^{x=b} \leq 0$ for all real valued functions satisfying BC, then there is no negative eigenvalue.

Proof

We first show that for any $g(x)$, $f(x)$ in (a,b)

$$\int_a^b f''(x)g(x)dx = -\int_a^b f'(x)g'(x)dx + f'g\Big|_a^b$$

$\boxed{\text{Green's first identity}}$ (or integration by parts

Now take $g(x) = f(x) = X(x)$ eigenfunction

$$\int_a^b X''(x)\,X(x)\,dx = -\int_a^b X'(x)^2\,dx + X'X\Big|_a^b$$

$$\|$$

$$-\lambda \underset{\overset{\vee}{0}}{\int_a^b X(x)^2\,dx} = -\underset{\overset{\wedge}{0}}{\int_a^b X'(x)^2\,dx} + \underset{\overset{\wedge}{0}}{X'X\Big|_a^b}$$

If $\lambda < 0$ then LHS > 0 which is a contradiction! Q.E.D.

Remark: For which standard (BC) does (★★) hold?

a) **Dirichlet**: $\underset{\overset{\|}{0}}{f(b)}\,f'(b) - \underset{\overset{\|}{0}}{f(a)}\,f'(a) = 0$ ✓

b) Neumann: $\overset{\|}{0} \qquad \overset{\|}{0} \quad = 0$ ✓

c) <u>Periodic Condition</u>: $f(b)f'(b) - f(a)f'(a)$

$= f(b)f'(b) - f(b)f'(b) = 0$

d) <u>Robin condition</u>: $u_x(a,t) - a_a u(a,t) = 0$

$u_x(b,t) + a_b u(a,t) = 0$

Case $a_a < 0$, $a_b > 0$ $\quad a_a + a_b > 0$

$a_a + a_b < -a_a a_b l \quad l = (b-a)$

} we know then there is a negative eigenvalue

So the condition (**) must fail!

~~[illegible]~~

~~[illegible]~~

<u>Exercise</u>: Consider the operator

$$A = \frac{d^4}{dx^4} \quad \text{with (BC)} \quad \begin{cases} X(0) = X(l) = 0 \\ X''(0) = X''(l) = 0 \end{cases}$$

Prove that there are <u>no</u> ~~[illegible]~~ <u>negative</u> <u>eigenvalues</u>.

By the ~~FOR~~ first Green's identity

$$\int_a^b f''(x) g(x) \, dx = -\int_a^b f'(x) g'(x) \, dx + f'g\Big|_a^b$$

~~f(x)~~ let $X(x)$ be an eigenfunction and λ eigenvalue

$f = X''(x) \quad g(x) = X(x)$

⑨

$$\int_0^\ell X^{(4)}(x)\,X(x)\,dx = -\int_0^\ell X'''(x)X'(x)\,dx + X'''(x)X(x)\Big|_0^\ell$$

$$+\lambda \underset{\overset{\vee}{0}}{\int_0^\ell X(x)^2\,dx} = -X''\overset{=0}{X'}(x)\Big|_0^\ell + \underset{\overset{\vee}{0}}{\int_0^\ell X''X''(x)\,dx}$$

$$+ \underset{\overset{\shortparallel}{0}}{X'''(x)X(x)\Big|_0^\ell}$$

So $\lambda > 0$!

Infinite Series of functions: Summary

A series

$$\sum_{k=1}^{\infty} a_n \text{ converges} \iff$$

the sequence of partial sums

$$\{S_n\} = \left\{\sum_{k=1}^{n} a_n\right\} \text{ converges to a limit } S$$

and in that case $\sum_{k=1}^{\infty} a_n = S$.

$\lim_{n\to\infty} S_n = S$ means that

For any $\varepsilon > 0$ there exist $N = N(\varepsilon)$ s.t. for all $n \geq N$

$$|S_n - S| < \varepsilon.$$

If $\lim_{n\to\infty} S_n = \pm\infty$ (or does not exists) we say that $\sum_{k=1}^{\infty} a_n$ diverges

~~and if $\lim_{n\to\infty} S_n$ does not exists~~

- If $\sum_{k=1} a_k$ converges then $\lim_{k\to+\infty} a_n = 0$

The viceversa is not true in general:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty \text{ diverges but } a_n = \frac{1}{k} \xrightarrow[k\to\infty]{} 0!$$

- ~~If~~ Given $\sum_{n=1}^{\infty} a_n$, if $\sum_{k=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

a) ~~if $\sum_{k=1}^{\infty} a_n$ is abso~~

a) Absolute convergence $\Longrightarrow$ Convergence

$$\not\Longleftarrow \text{ in general}$$

- If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not then we say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent

Comparison test:

- If $|a_n| \le b_n$ for $n \ge N_0$ and $\sum_{n=N_0} b_n < \infty$ then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.
- If $\sum_{n=1}^{\infty} |a_n|$ diverges $\Longrightarrow$ $\sum_{k=1}^{\infty} b_k$ diverges