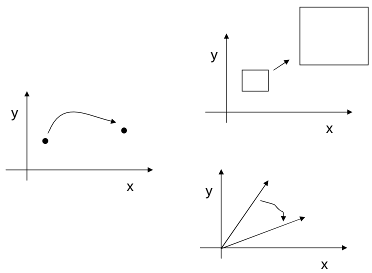
2D Transformations



2D Transformation

- Given a 2D object, transformation is to change the object's
  - Position (translation)
  - Size (scaling)
  - Orientation (rotation)
  - Shapes (shear)
- Apply a sequence of matrix multiplication to the object vertices

Point representation

- We can use a column vector (a 2x1 matrix) to represent a 2D point $\begin{vmatrix} x \\ y \end{vmatrix}$
- A general form of *linear* transformation can be written as:

$x' = ax + by + c$

$y' = dx + ey + f$

OR $\begin{vmatrix} X' \\ Y' \\ 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$

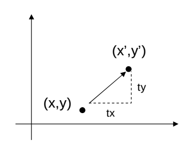
Translation

- Re-position a point along a straight line
- Given a point (x,y), and the translation distance (tx,ty)

The new point: (x', y')

$x' = x + tx$

$y' = y + ty$



OR $P' = P + T$ where $P' = \begin{vmatrix} x' \\ y' \end{vmatrix}$ $p = \begin{vmatrix} x \\ y \end{vmatrix}$ $T = \begin{vmatrix} tx \\ ty \end{vmatrix}$

3x3 2D Translation Matrix

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} tx \\ ty \end{vmatrix}$$

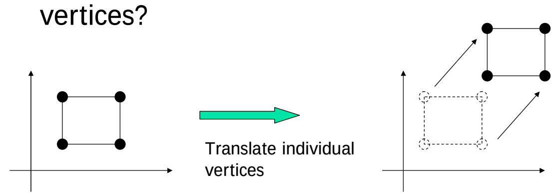
Use 3 x 1 vector

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

- Note that now it becomes a matrix-vector multiplication

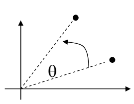
Translation

- How to translate an object with multiple vertices?

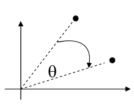


2D Rotation

- Default rotation center: Origin (0,0)



$\theta > 0$: Rotate counter clockwise



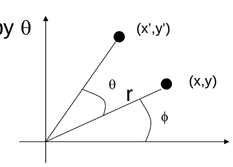
$\theta < 0$: Rotate clockwise

Rotation

(x,y) -> Rotate *about the origin* by θ

→ (x', y')

How to compute (x', y') ?



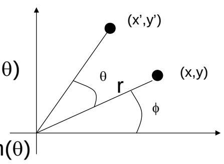
$x = r \cos(\phi) \quad y = r \sin(\phi)$

$x' = r \cos(\phi + \theta) \quad y = r \sin(\phi + \theta)$

Rotation

$x = r\cos(\phi) \quad y = r\sin(\phi)$

$x' = r\cos(\phi+\theta) \quad y = r\sin(\phi+\theta)$



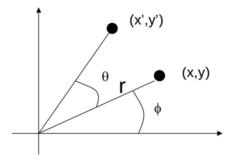
$x' = r\cos(\phi+\theta)$
$= r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)$
$= x\cos(\theta) - y\sin(\theta)$

$y' = r\sin(\phi+\theta)$
$= r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta)$
$= y\cos(\theta) + x\sin(\theta)$

Rotation

$x' = x\cos(\theta) - y\sin(\theta)$

$y' = y\cos(\theta) + x\sin(\theta)$



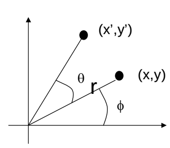
Matrix form?

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$

3 x 3?

3x3 2D Rotation Matrix

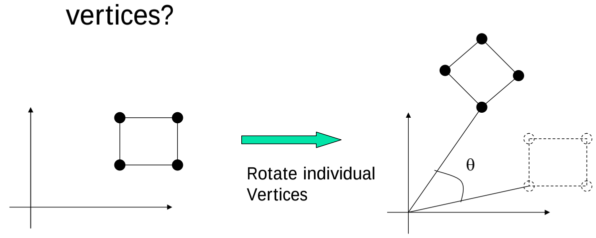
$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$



$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Rotation

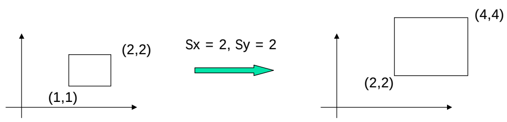
- How to rotate an object with multiple vertices?



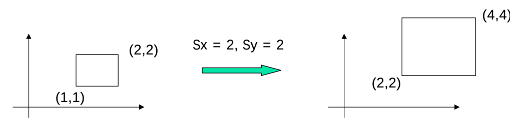
2D Scaling

Scale: Alter the size of an object by a scaling factor (Sx, Sy), i.e.

$$x' = x \cdot Sx \quad y' = y \cdot Sy \;\Rightarrow\; \begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} Sx & 0 \\ 0 & Sy \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$



2D Scaling



- Not only the object size is changed, it also moved!!
- Usually this is an undesirable effect
- We will discuss later (soon) how to fix it

3x3 2D Scaling Matrix

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} Sx & 0 \\ 0 & Sy \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$

$$\Downarrow$$

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Put it all together

- Translation: $\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} tx \\ ty \end{vmatrix}$
- Rotation: $\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} * \begin{vmatrix} x \\ y \end{vmatrix}$
- Scaling: $\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} Sx & 0 \\ 0 & Sy \end{vmatrix} * \begin{vmatrix} x \\ y \end{vmatrix}$

Or, 3x3 Matrix representations

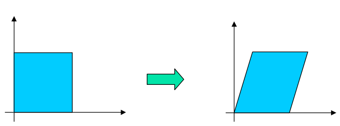
- Translation: $\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$
- Rotation: $\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$
- Scaling: $\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$

Why use 3x3 matrices?

Why use 3x3 matrices?

- So that we can perform all transformations using matrix/vector multiplications
- This allows us to *pre-multiply* all the matrices together
- The point (x,y) needs to be represented as (x,y,1) -> this is called Homogeneous coordinates!

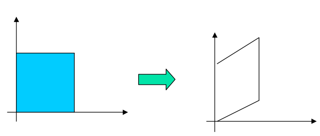
Shearing



- Y coordinates are unaffected, but x cordinates are translated linearly with y
- That is:
  - y' = y
  - x' = x + y * h

$$\begin{vmatrix} x \\ y \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Shearing in y



$$\begin{vmatrix} x \\ y \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ g & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

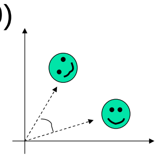
Interesting Facts:

- A 2D rotation is three shears
- Shearing will not change the area of the object
- Any 2D shearing can be done by a rotation, followed by a scaling, and followed by a rotation

Rotation Revisit

- The standard rotation matrix is used to rotate about the origin (0,0)

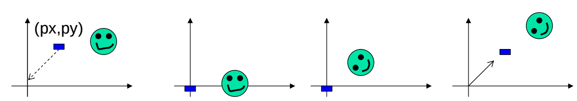
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$



- What if I want to rotate about an arbitrary center?

Arbitrary Rotation Center

- To rotate about an arbitrary point P (px,py) by θ:
  - Translate the object so that P will coincide with the origin: T(-px, -py)
  - Rotate the object: R(θ)
  - Translate the object back: T(px,py)



Arbitrary Rotation Center

- Translate the object so that P will coincide with the origin: T(-px, -py)
- Rotate the object: R(θ)
- Translate the object back: T(px,py)

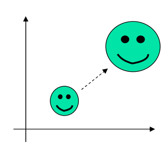
- Put in matrix form: T(px,py) R(θ) T(-px, -py) * P

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & px \\ 0 & 1 & py \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & -px \\ 0 & 1 & -py \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Scaling Revisit

- The standard scaling matrix will only anchor at (0,0)

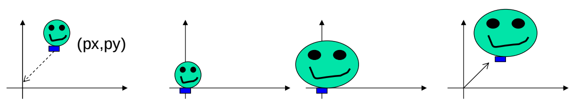
$$\begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$



- What if I want to scale about an arbitrary pivot point?

Arbitrary Scaling Pivot

- To scale about an arbitrary pivot point P (px,py):
  - Translate the object so that P will coincide with the origin: T(-px, -py)
  - Rotate the object: S(sx, sy)
  - Translate the object back: T(px,py)



Affine Transformation

- Translation, Scaling, Rotation, Shearing are all affine transformation
- Affine transformation – transformed point P' (x',y') is a linear combination of the original point P (x,y), i.e.

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} m11 & m12 & m13 \\ m21 & m22 & m23 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

- Any 2D affine transformation can be decomposed into a rotation, followed by a scaling, followed by a shearing, and followed by a translation.

  Affine matrix = translation x shearing x scaling x rotation

Composing Transformation

- Composing Transformation – the process of applying several transformation in succession to form one overall transformation
- If we apply transform a point P using M1 matrix first, and then transform using M2, and then M3, then we have:

(M3 x (M2 x (M1 x P))) = M3 x M2 x M1 x P

(pre-multiply) — M3 x M2 x M1 → M

Composing Transformation

- Matrix multiplication is associative

  M3 x M2 x M1 = (M3 x M2) x M1 = M3 x (M2 x M1)
- Transformation products may not be commutative

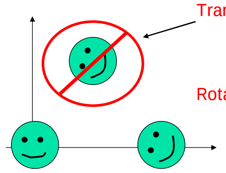
  A x B != B x A
- Some cases where A x B = B x A

A	B
translation	translation
scaling	scaling
rotation	rotation
uniform scaling	rotation

(sx = sy)

Transformation order matters!

- Example: rotation and translation are not commutative



Translate (5,0) and then Rotate 60 degree

OR

Rotate 60 degree and then translate (5,0)??

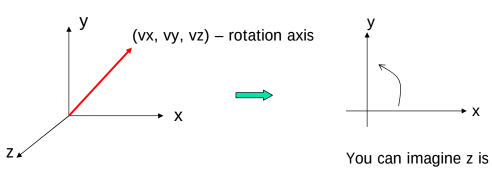
Rotate and then translate !!

How OpenGL does it?

- OpenGL's transformation functions are meant to be used in 3D
- No problem for 2D though – just ignore the z dimension
- Translation:
  - glTranslatef(d)(tx, ty, tz) -> glTranslatef(d)(tx,ty,0)   for 2D

How OpenGL does it?

- Rotation:
  - glRotatef(d)(angle, vx, vy, vz) -> glRotatef(d)(angle, 0,0,1)   for 2D



You can imagine z is pointing out of the slide

OpenGL Transformation Composition

- A global modeling transformation matrix (GL_MODELVIEW, called it M here)
  glMatrixMode(GL_MODELVIEW)
- The user is responsible to reset it if necessary
  glLoadIdentity()
  -> M = 1 0 0
  0 1 0
  0 0 1

OpenGL Transformation Composition

- Matrices for performing user-specified transformations are multiplied to the model view global matrix
- For example,

glTranslated(1,1 0); $M = M \times \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$

- All the vertices P defined within glBegin() will first go through the transformation (modeling transformation)

$$P' = M \times P$$

Transformation Pipeline