Linear Systems

Pivoting in Gaussian Elim.

CSE 541
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Limitations of Gaussian Elimation

- The ***naïve*** implementation of Gaussian Elimination is not robust and can suffer from severe round-off errors due to:
  - Dividing by zero
  - Dividing by small numbers and adding.
- Both can be solved with ***pivoting***

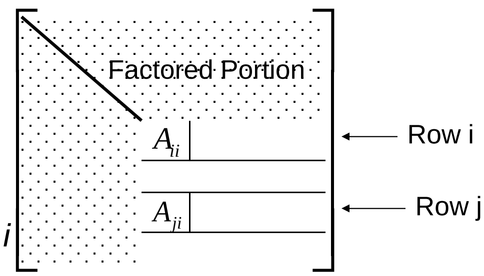
Partial Pivoting

- What if at step *i*, $A_{ii} = 0$?
- Simple Fix:

  If $A_{ii} = 0$

  Find $A_{ji} \neq 0$ $j > i$

  Swap Row *j* with *i*



Example – Partial Pivoting

$$\begin{bmatrix} 1.25\cdot10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

Forward Elimination

$$\begin{bmatrix} 1.25\cdot10^{-4} & 1.25 \\ 0 & 12.5-1.25\cdot10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75-6.25\cdot10^{5} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

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Rounded to 3 digits

$$\begin{bmatrix} 1.25\cdot10^{-4} & 1.25 \\ 0 & -1.25\cdot10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ -6.25\cdot10^{5} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3digits} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Better Pivoting

- Partial Pivoting to mitigate round-off error

$$\text{If } |A_{ii}| < \max_{j>i} |A_{ji}|$$

$$\text{Swap row } i \text{ with } \arg(\max_{j>i} |A_{ij}|)$$

Avoids Small Multipliers

- Adds an **O**(*n*) search.

Partial Pivoting

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

swap

$$\begin{bmatrix} 12.5 & 12.5 \\ 1.25\cdot 10^{-4} & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

Forward Elimination

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 - 12.5\cdot 10^{-5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 - 75\cdot 10^{-5} \end{bmatrix}$$

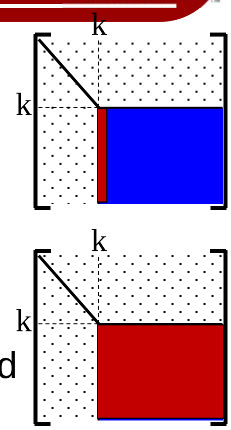
Rounded to 3 digits

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3digits} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Pivoting strategies

- Partial Pivoting:
  - Only row interchange
- Complete (Full) Pivoting
  - Row and Column interchange
- Threshold Pivoting
  - Only if prospective pivot is found to be smaller than a certain threshold



Pivoting With Permutations

- Adding permutation matrices in the mix:

$$M_{n-1}P_{n-1}M_{n-2}P_{n-2}\cdots M_1P_1Ax = M_{n-1}P_{n-1}M_{n-2}P_{n-2}\cdots M_1P_1b$$

- However, in Gaussian Elimination we will only swap rows or columns below the current pivot point. This implies a global reordering of the equations will work:

$$MPAx = MPb$$
$$MA'x = b'$$

Pivoting

- Again, the pivoting is strictly a function of the matrix A, so once we determine P it is trivial to apply it to many problems b_k.
- For LU factorization we have:
  - LU = PA
  - Ly = Pb
  - Ux = y