Numerical Integration

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Quadrature

- We talk in terms of Quadrature <u>Rules</u>
  - **1.** The process of making something square. **2.** *Mathematics* The process of constructing a square equal in area to a given surface. **3.** *Astronomy* A configuration in which the position of one celestial body is 90° from another celestial body, as measured from a third.
    - The American Heritage® Dictionary: Fourth Edition. 2000

Outline

- Definite Integrals
- Lower and Upper Sums
  - Reimann Integration or Reimann Sums
- Uniformly-spaced samples
  - Trapezoid Rules
  - Romberg Integration
  - Simpson’s Rules
  - Adaptive Simpson’s Scheme
- Non-uniformly spaced samples
  - Gaussian Quadrature Formulas

Motivation

What does an integral represent?

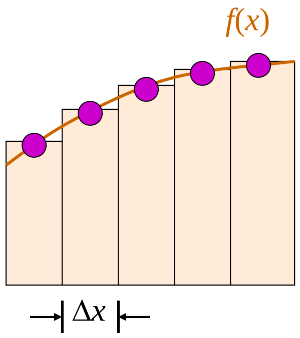
$$\int_a^b f(x)dx = \text{area} \qquad \int_c^d \int_a^b f(x)dxdy = \text{volume}$$

Basic definition of an integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$

sum of height × width



Motivation

- Evaluate the integral, $I = \int_a^b f(x)dx$ without doing the calculation analytically.
- Necessary when either:
  - Integrand is too complicated to integrate analytically

    $$\int_0^2 \frac{2+\cos(1+\sqrt{x})}{\sqrt{1+0.5x}} e^{0.5x} dx$$

  - Integrand is not precisely defined by an equation, i.e., we are given a set of data (x_i, y_i), $i = 1, 2, 3, \ldots, n$

Reimann Integral Theorem

- Integration is a summing process. Thus virtually all numerical approximations can be represented by

$$I = \int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + E_t$$

in which w_i are the weights, x_i are the sampling points, and E_t is the truncation error

- Valid for any function that is continuous on the closed and bounded interval of integration.

Partitioning the Integral

- The most common numerical integration formula is based on equally spaced data points.

$$\int_{x_0}^{x_n} f(x)dx$$

- Divide [x_0, x_n] into *n* ***intervals*** ($n \geq 1$)

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x) + \int_{x_1}^{x_2} f(x) + \cdots + \int_{x_{n-1}}^{x_n} f(x)$$

Upper Sums

- Assume that $f(x)>0$ everywhere.
- If within each interval, we could determine the maximum value of the function, then we have:

$$\int_{x_0}^{x_n} f(x) \leq \sum_{i=0}^{n-1} M_i \left(x_{i+1} - x_i\right)$$

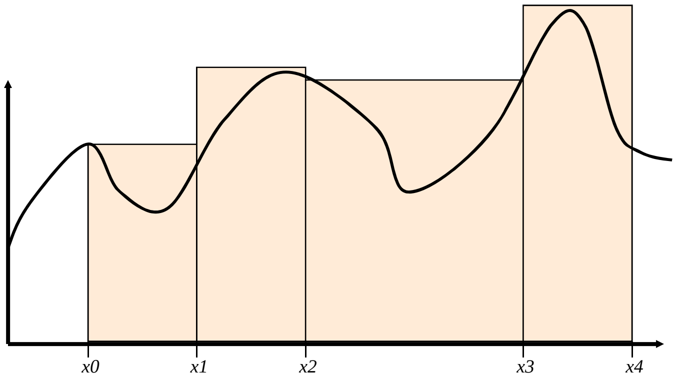
- where

$$M_i = \sup\left\{ f(x) : x_i \leq x \leq x_{i+1} \right\}$$

Supremum - least upper bound

Upper Sums

- Graphically:



Lower Sums

- Likewise, still assuming that *f(x)>0* everywhere.
- If within each interval, we could determine the minimum value of the function, then we have:

$$\int_{x_0}^{x_n} f(x) \ge \sum_{i=0}^{n-1} m_i \left(x_{i+1} - x_i \right)$$

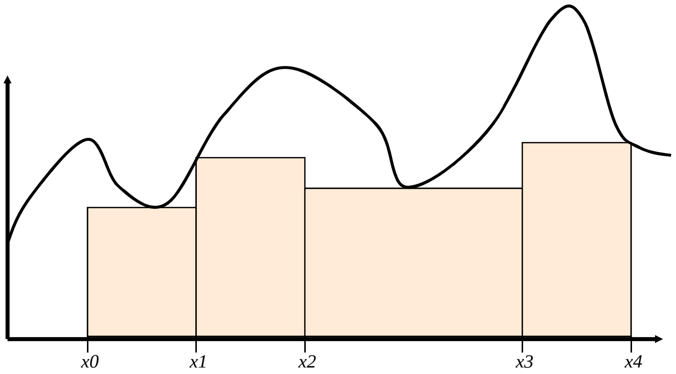
- where

$$m_i = \inf \left\{ f(x) : x_i \le x \le x_{i+1} \right\}$$

Infimum - greatest lower bound

Lower Sums

- Graphically



Finer Partitions

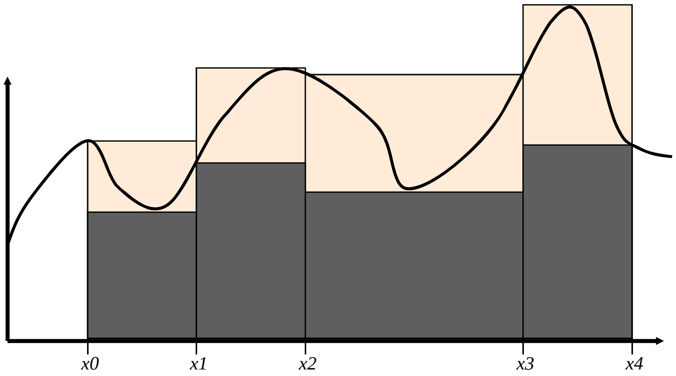
- We now have a bound on the integral of the function for some partition ($x_0,..,x_n$):

$$\sum_{i=0}^{n-1} m_i \left(x_{i+1}-x_i\right) \le \int_{x_0}^{x_n} f(x) \le \sum_{i=0}^{n-1} M_i \left(x_{i+1}-x_i\right)$$

- As $n \to \infty$, one would assume that the sum of the upper bounds and the sum of the lower bounds approach each other.
- This is the case for most functions, and we call these Riemann-integrable functions.

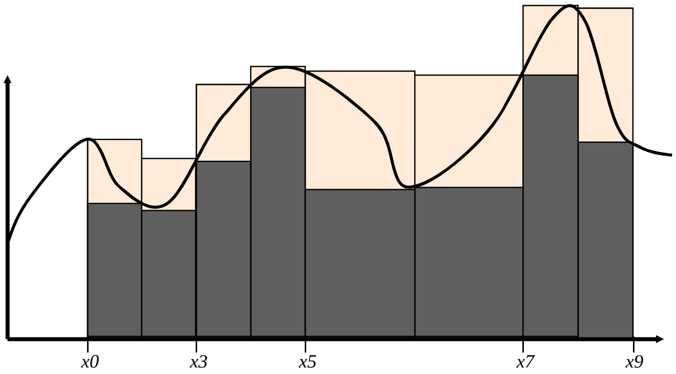
Bounding the Integral

- Graphically



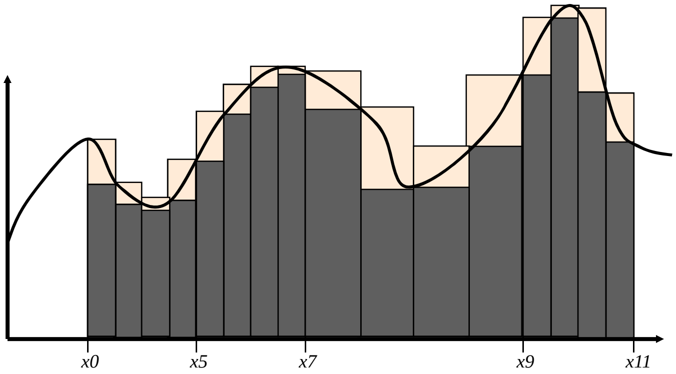
Bounding the Integral

- Halving each interval *(much like Lab1)*:



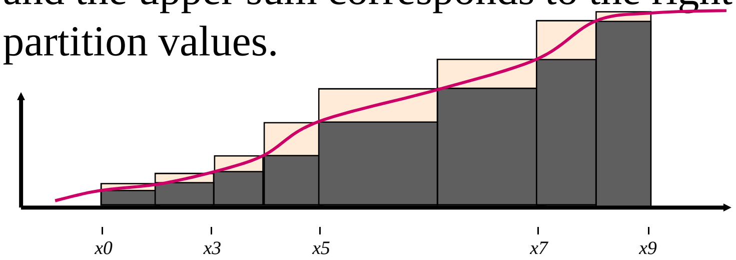
Bounding the Integral

- One more time:



Monotonic Functions

- Note that if a function is monotonically increasing (or decreasing), then the lower sum corresponds to the left partition values, and the upper sum corresponds to the right partition values.



Lab1 and Integration

- Thinking back to lab1, what were the limits or the integration?
- Is the sin function monotonic on this interval?
- Should the Reiman sum be an upper or lower sum?

Polynomial Approximation

- Rather than search for the maximum or minimum, we replace *f(x)* with a known and simple function.
- Within each interval we approximate $f(x)$ by an $m^{\underline{\text{th}}}$ order polynomial.

$$p_m(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

Newton-Cotes Formulas

- The *m*’s (order of the polynomials) may be the same or different.

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_{0+m_1}} p_{m_1}(x)dx + \int_{x_{0+m_1}}^{x_{0+m_1+m_2}} p_{m_2}(x)dx + \ldots + \int_{x_{n-m_n}}^{x_n} p_{m_n}(x)dx$$

- Different choices for *m*’s lead to different formulas:

m	Polynomial	Formula	Error
1	linear	Trapezoid	$O(h^2)$
2	quadratic	Simpson's 1/3	$O(h^4)$
3	cubic	Simpson's 3/8	$O(h^4)$

Trapezoid Rule

- Simplest way to approximate the area under a curve – using **first order polynomial** (a straight line)
- Using Newton’s form of the interpolating polynomial:

$$p_1(x) = f(a) + \frac{f(b) - f(a)}{b-a}(x-a)$$

- Now, solve for the integral:

$$I = \int_a^b f(x)dx \approx \int_a^b p_1(x)dx$$

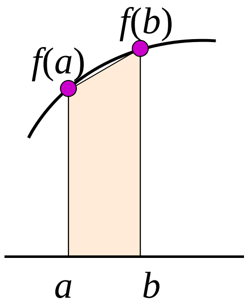
Trapezoid Rule

$$I \approx \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

Trapezoid Rule

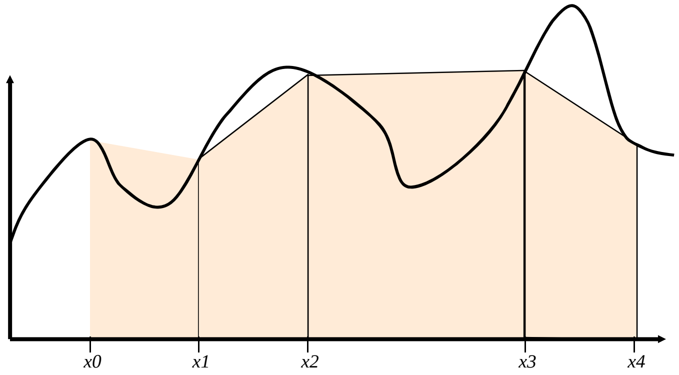
$$\boxed{I \approx \frac{(b - a)}{2}[f(a) + f(b)]}$$

$I \approx$ width × average height



Trapezoid Rule

- Improvement?



Trapezoid Rule Error

- The integration error is:

$$E_t = -\frac{1}{12} f''(\xi)h^3 = -\frac{(b-a)}{12} f''(\xi)h^2 \qquad O(h^3)$$

- Where $h = b - a$ and ξ is an unknown point where $a < \xi < b$ (intermediate value theorem)
- You get exact integration if the function, f, is linear ($f'' = 0$)

Example

Integrate from $f(x) = e^{-x^2}$ $a = 0$ to $b = 2$

Use trapezoidal rule:

$$\begin{aligned} I &= \int_0^2 e^{-x^2} dx \\ &\approx \frac{(b-a)}{2}\left[f(a) + f(b)\right] = \frac{(2-0)}{2}\left[f(2) + f(0)\right] \\ &= 1 \times (e^{-4} + e^{0}) = 1.0183 \end{aligned}$$

Example

Estimate error: $E_t = -\frac{1}{12} f''(\xi) h^3$

Where $h = b - a$ and $a < \xi < b$

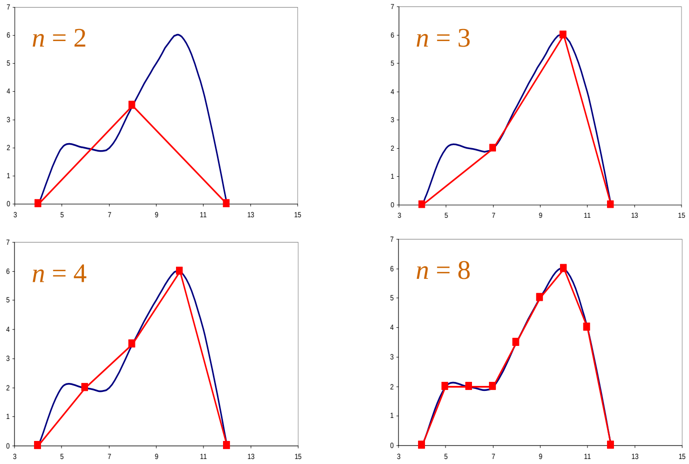
Don’t know ξ - use average value

$$f''(x) = (-2 + 4x^2)e^{-x^2} \qquad f''(0) = -2$$

$$h = 2 - 0 = 2 \qquad f''(2) = 0.2564$$

$$E_t \approx E_a = -\frac{2^3}{12}\frac{[f''(0) + f''(2)]}{2} = 0.58$$

More intervals, better result [error ~ $O(h^2)$]



Composite Trapezoid Rule

- If we do multiple intervals, we can avoid duplicate function evaluations and operations:
- Use n+1 equally spaced points.
- Each interval has: $h = \frac{b-a}{n}$
- Break up the limits of integration and expand.

$$I = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{b-h}^{b} f(x)dx$$

Composite Trapezoid Rule

- Substituting the trapezoid rule for each integral.

$$I = \int_a^{a+h} f(x)\,dx + \int_{a+h}^{a+2h} f(x)\,dx + \quad \ldots \quad + \int_{b-h}^{b} f(x)\,dx$$

$$= \frac{(a+h-a)}{2}\left[f(a)+f(a+h)\right] + \frac{(a+2h-a-h)}{2}\left[f(a+h)+f(a+2h)\right]$$

$$+ \ldots. + \frac{(b-b+h)}{2}\left[f(b-h)+f(b)\right]$$

- Results in the Composite Trapezoid Formula:

$$\boxed{I = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right]}$$

Composite Trapezoid Rule

- Think of this as the *width* times the average *height*.

$$I = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b) \right]$$

$$= \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)}{2n}}_{\text{Average height}}$$

Error

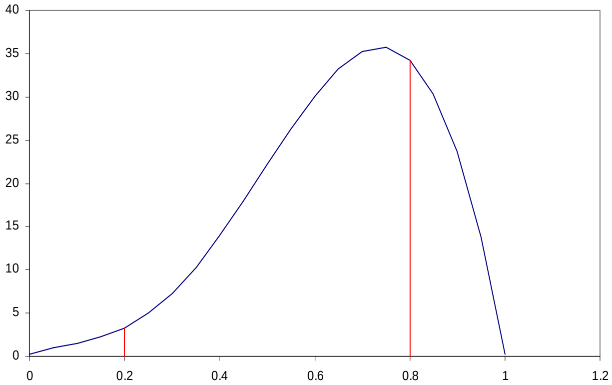
- The error can be estimated as:

$$\boxed{E_a = \frac{(b-a)h^2}{12}\overline{f}'' = \frac{(b-a)^3}{12n^2}\overline{f}''} \quad O(h^2)$$

- Where, $\overline{f}''$ is the average second derivative.
- If n is doubled, $h \rightarrow h/2$ and $E_a \rightarrow E_a/4$
- Note, that the error is dependent upon the *width* of the area being integrated.

Example

- Integrate: $f(x) = 0.3 + 20x - 140x^2 + 730x^3 - 810x^4 + 200x^5$
- from a=0.2 to b=0.8



Example

- A single application of the Trapezoid rule.

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

$$= (0.8-0.2)\frac{34.22+3.81}{2}$$

$$= 11.26$$

- Error: $E_t = -\frac{1}{12}f''(\xi)(b-a)^3$

Example

- We don't know ξ so approximate with average f''

$$f'(x) = 20 - 280x + 2190x^2 - 3240x^3 + 1000x^4$$

$$f''(x) = -280 + 4380x - 9720x^2 + 4000x^3$$

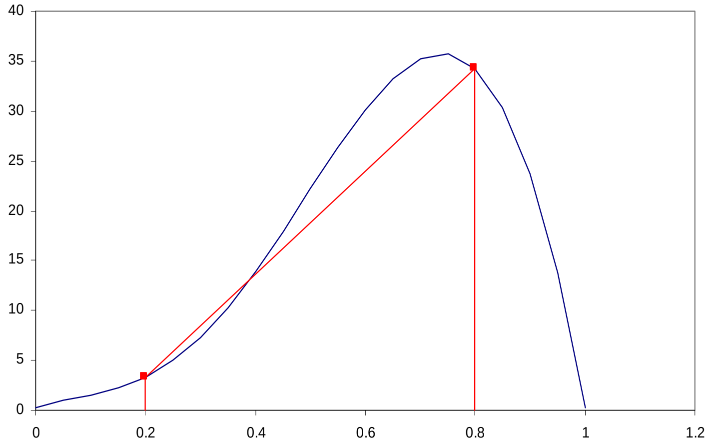
$$\begin{aligned}\bar{f}''(x) &= \frac{\int_{0.2}^{0.8} f''\,dx}{0.8 - 0.2}\\ &= \frac{f'(0.8) - f'(0.2)}{0.8 - 0.2} = -131.6\end{aligned}$$

Example

- The error can thus be estimated as:

$$\begin{aligned}E_t &= \frac{(b-a)h^2}{12}\bar{f}'' = \frac{(b-a)^3}{12n^2}\bar{f}''\\ &= -\frac{1}{12}(-131.6)(0.8-0.2)^3 = 2.37\end{aligned}$$

True value of integral is 12.82. Trapezoid rule is 11.26 - within approx error - E_t is 12%



Using Three Intervals

- Use intervals (0.2,0.4),(0.4,0.6),(0.6,0.8):
  - (n = 3, h = 0.2)

$$I = (b-a)\frac{f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)}{2n}$$

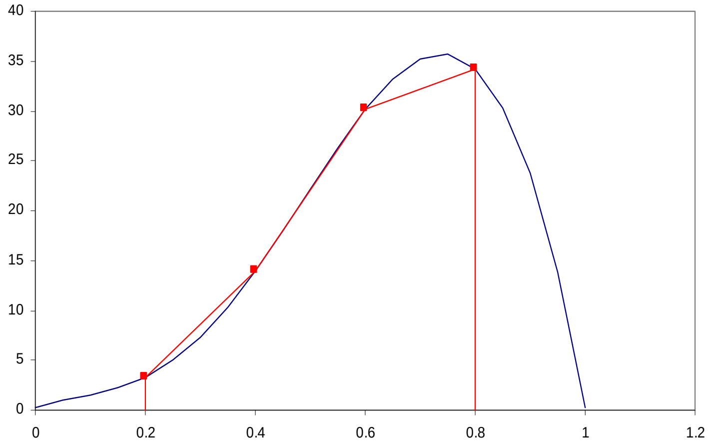
$$= (0.8-0.2)\frac{f(0.2)+2[f(0.4)+f(0.6)]+f(0.8)}{(2)(3)}$$

$$= 0.6\frac{3.31+2(13.93+30.16)+34.22}{6}$$

$$= 12.57$$

True value of integral is 12.82

E_t is now 2%



Using Six Intervals

- Use intervals (0.2,0.3),(0.3,0,4), etc.
  - $(n = 6, h = 0.1)$

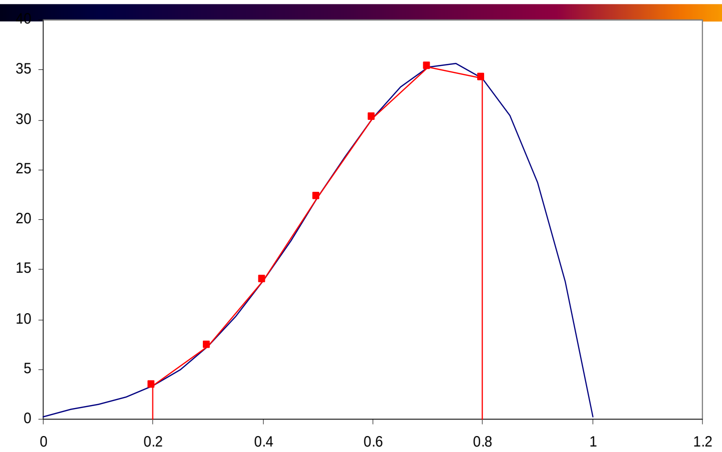
$$I = (0.8-0.2)\frac{f(0.2)+2[f(0.3)+f(0.4)+f(0.5)+f(0.6)+f(0.7)]+f(0.8)}{(2)(6)}$$

$$= 0.6\frac{3.31+2(7.34+13.93+22.18+30.16+35.22)+34.22}{12}$$

$$= 12.76$$

True value of integral is 12.82

E_t is now 0.5%



Multi-dimensional Integration

- Consider a two-dimensional case.

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)\,dxdy &\approx \int_0^1 \sum_{i=0}^n A_i f\left(\frac{i}{n},y\right)dy\\ &= \sum_{i=0}^n A_i \int_0^1 f\left(\frac{i}{n},y\right)dy\\ &\approx \sum_{i=0}^n A_i \sum_{j=0}^n A_j f\left(\frac{i}{n},\frac{j}{n}\right)\\ &= \sum_{i=0}^n \sum_{j=0}^n A_i A_j f\left(\frac{i}{n},\frac{j}{n}\right)\end{aligned}$$

Multi-dimensional Integration

- For the Trapezoid Rule, this leads to weights in the following pattern:

1	1	2	2	2	2	2	1
2	2	4	4	4	4	4	2
2	2	4	4	4	4	4	2
2	2	4	4	4	4	4	2
2	2	4	4	4	4	4	2
2	2	4	4	4	4	4	2
1	1	2	2	2	2	2	1
	1	2	2	2	2	2	1

$$A_{ij} = \frac{1}{4n^2}\begin{cases} 1 & i \in \{0, n\} & j \in \{0, n\} \\ 2 & i \in [1, \ldots, n-2] & j \in \{1, n-1\} \\ 2 & i \in \{1, n-1\} & j \in [1, \ldots, n-1] \\ 4 & i \in [1, \ldots, n-2] & j \in [1, \ldots, n-2] \end{cases}$$

Multi-dimensional Integration

- If we use the weights from the Trapezoid rule, the error is still $O(h^2)$.
- However, there are now n^2 function evaluations.
  - Equally-spaced samples on a square region.

Multi-dimensional Integration

- In general, given k dimensions, we have $N = n^k$ function evaluations:

$$O\left(h^2\right) = O\left(n^{-2}\right) = O\left(\left(n^k\right)^{-\frac{2}{k}}\right) = O\left(N^{-\frac{2}{k}}\right)$$

- If the dimension is high, this leads to a significant amount of additional work in going from $h \rightarrow h/2$.
  - Remember this for Monte-Carlo Integration.

Reducing the Error

- To improve the estimate of the integral, we can either:
  - Add more intervals
  - Use a higher order polynomial
  - Use Richardson Extrapolation to examine the limit as h→0.
    - Called Romberg Integration

Adding More Intervals

- If we have an estimate for one value of *h*, do we need to recompute everything for a value of *h*/2?

$$\boxed{I_h = \frac{h}{2}\left[f\left(a\right) + 2\sum_{i=1}^{n-1} f\left(a+ih\right) + f\left(b\right) \right]}$$

Adding More Intervals

- This is called the **Recursive Trapezoid Rule** in the book.
- We have $n \rightarrow 2n$ and $h \rightarrow h/2$.

$$I_{\frac{h}{2}} = \frac{h}{4}\left[f(a) + 2\sum_{i=1}^{2n-1} f\left(a + i\frac{h}{2}\right) + f(b)\right]$$

$$= \frac{h}{4}\left[f(a) + 2\sum_{i=1}^{n-1} f\left(a+ih\right) + 2\sum_{i=0}^{n-1} f\left(a+ih+\frac{h}{2}\right) + f(b)\right]$$

$$= \frac{I_h}{2} + \frac{h}{4}\left[2\sum_{i=0}^{n-1} f\left(a+ih+\frac{h}{2}\right)\right]$$

Recall Richardson Extrapolation

- Given two numerical estimates obtained using different *h*'s, compute higher-order estimate
- Starting with a step size h_1, the exact value is

$$A = A(h_1) + O(h_1^{\,n})$$

- Suppose we reduce step size to h_2

$$A = A(h_2) + O(h_2^{\,n})$$

Richardson Extrapolation

- Multiplying the 2nd eqn by $(h_1/h_2)^n$ and subtracting from the 1st eqn:

$$A = \frac{\left(\frac{h_1}{h_2}\right)^n A(h_2) - A(h_1)}{\left(\frac{h_1}{h_2}\right)^n - 1}$$

- The new error term is generally $O(h_1^{\,n+1})$ or $O(h_1^{\,n+2})$.

Richardson Extrapolation

- The true integral value can be written

$$I = I(h) + E(h)$$

- This is true for any iteration

$$I = I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

Richardson Extrapolation

- For example: Using (n = 2)

$$E \approx ch^2 \overline{f}''$$ [error = $O(h^2)$]

order of error in trapezoidal rule

- where c is a constant
- Therefore:

$$\frac{E(h_1)}{E(h_2)} \approx \frac{h_1^2}{h_2^2}$$

Richardson Extrapolation

- This leads to:

$$E\left(h_1\right) \approx E\left(h_2\right)\frac{h_1^{\ 2}}{h_2^{\ 2}}$$

- For integration, we have:

$$I\left(h_1\right) + E\left(h_2\right)\frac{h_1^{\ 2}}{h_2^{\ 2}} \approx I\left(h_2\right) + E\left(h_2\right)$$

Richardson Extrapolation

- Solving for $E(h_2)$:

$$E\left(h_2\right) \approx \frac{I\left(h_1\right) - I\left(h_2\right)}{1 - \frac{h_1^{\ 2}}{h_2^{\ 2}}}$$

- And plugging back into the estimated integral.

$$I = I\left(h_2\right) + E\left(h_2\right)$$

Richardson Extrapolation

- Leads to:

$$I \approx I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}\left[I(h_2) - I(h_1)\right]$$

- Letting $h_2 = h_1/2$

$$I \approx I(h_2) + \frac{1}{2^2 - 1}\left[I(h_2) - I(h_1)\right]$$

$$\boxed{I \approx \frac{4}{3}I(h_2) - \frac{1}{3}I(h_1)}$$

Romberg Integration

- We combined two $O(h^2)$ estimates to get an $O(h^4)$ estimate.
- Can also combine two $O(h^4)$ estimates to get an $O(h^6)$ estimate.

$$I \approx \frac{16}{15}I(h_m) - \frac{1}{15}I(h_l)$$

Romberg Integration

- Greater weight is placed on the more accurate estimate.
- Weighting coefficients sum to unity
  - i.e, (16-1)/15=1
- Can continue, by combining two $O(h^6)$ estimates to get an $O(h^8)$ estimate.

$$I \approx \frac{64}{63} I(h_m) - \frac{1}{63} I(h_l)$$

Romberg Integration

- General pattern is called Romberg Integration

$$I_{j,k+1} \approx \frac{4^k I_{j,k} - I_{j-1,k}}{4^k - 1} = I_{j,k} + \frac{1}{4^k - 1}\left(I_{j,k} - I_{j-1,k}\right)$$

  - j : level of subdivision, j+1 has more intervals.
  - k : level of integration, k = 1 is original trapezoid estimate [$O(h^2)$], k = 2 is improved [$O(h^4)$], etc.

Romberg Integration

- For example, j = 1, k = 1 leads to

$$I_{1,1} = \frac{4I_{1,0} - I_{0,0}}{3} \approx \frac{4}{3} I\left(\frac{h_1}{2}\right) - \frac{1}{3} I\left(h_1\right)$$

Example

- Consider the function:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

- Integrate from a = 0 to b = 0.8
- Using the trapezoidal rule yields the following results:

Example

- Trapezoid Rules:

	intervals	h	$k = 0$ Integral	$k = 1$
$j = 0$	1	0.8	0.1728	
$j = 1$	2	0.4	1.0688	
$j = 2$	4	0.2	1.4848	

$$I = \frac{4}{3}(1.0688) - \frac{1}{3}(0.1728) = 1.3674667$$ (j=1, k=1)

Exact integral is 1.64053334

Example

segments	h	$k = 0$ $O(h^2)$	$k = 1$ $O(h^4)$
1	0.8	0.1728	
2	0.4	1.0688	1.3674667
4	0.2	1.4848	

$$I = \frac{4}{3}(1.4848) - \frac{1}{3}(1.0688) = 1.62346667$$ (j=2, k=1)

Exact integral is 1.64053334

Example

k →, j ↓

		$k = 1$	$k = 2$
segments	h	$O(h^2)$	$O(h^4)$
1	0.8	0.1728	
2	0.4	1.0688	1.3674667
4	0.2	1.4848	1.62346667

(j=2, k=2) $$I=\frac{16}{15}(1.62346667)-\frac{1}{15}(1.3674667)=1.64053334$$

Exact integral is 1.64053334

Example

k →, j ↓

		$k = 1$	$k = 2$	$k = 3$
segments	h	$O(h^2)$	$O(h^4)$	$O(h^6)$
1	0.8	0.1728		
2	0.4	1.0688	1.3674667	
4	0.2	1.4848	1.62346667	1.64053334

Example

- Better and better results can be obtained by continuing this

k → $k = 3$

j ↓

segments	h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
1	0.8	0.1728			
2	0.4	1.0688	1.3674667		
4	0.2	1.4848	1.62346667	1.64053334	
8	0.1	??	??	??	??

$$I = \frac{64}{63}(??) - \frac{1}{63}(1.64053334) = ??$$

(j=3, k=3)

Romberg Integration

- Is this ***that*** significant?
- Consider the cost of computing the Trapezoid Rule for 1000 data points.
  - Refinement would lead to 2000 data points.
    - Implies an additional 1003 operations using the Recursive Trapezoid Rule.
    - Not to mention the 1000 (expensive) function evals.
  - Romberg Integration cost:
    - Three additional operations – no function evals!!!

Higher-Order Polynomials

- Recall:

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_{0+m_1}} p_{m_1}(x)dx + \int_{x_{0+m_1}}^{x_{0+m_1+m_2}} p_{m_2}(x)dx + \ldots + \int_{x_{n-m_n}}^{x_n} p_{m_n}(x)dx$$

m	Polynomial	Formula	Error
1	linear	Trapezoid	$O(h^2)$
2	quadratic	Simpson's 1/3	$O(h^4)$
3	cubic	Simpson's 3/8	$O(h^4)$

Simpson's 1/3 Rule

- If we use a 2nd order polynomial (need 3 points or 2 intervals):
  - Lagrange form. $\left(x_1 = \frac{x_0 + x_2}{2} \right)$

$$I = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

Simpson's 1/3 Rule

- **Requiring** equally-spaced intervals:

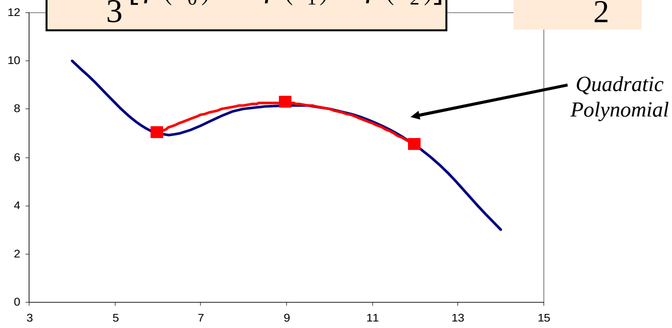
$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_0-h)(x-x_0-2h)}{-h(-2h)} f(x_0) + \frac{(x-x_0)(x-x_0-2h)}{(h)(-h)} f(x_1) + \frac{(x-x_0)(x-x_0-h)}{(2h)(h)} f(x_2) \right] dx$$

Simpson's 1/3 Rule

- Integrate and simplify:

$$\boxed{I \approx \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]}$$

$$h = \frac{b-a}{2}$$



Simpson's 1/3 Rule

- If we use $a = x_0$ and $b = x_2$, and $x_1 = (b+a)/2$

$$I \approx \underbrace{(b-a)}_{\text{width}}\underbrace{\frac{f(x_0)+4f(x_1)+f(x_2)}{6}}_{\text{average height}}$$

Simpson's 1/3 Rule

- Error for Simpson's 1/3 rule

$$E_t = -\frac{h^5}{90}f^{(4)}(\xi) = -\frac{(b-a)^5}{2880}f^{(4)}(\xi) \qquad O(h^5)$$

$$h = \frac{b-a}{2}$$

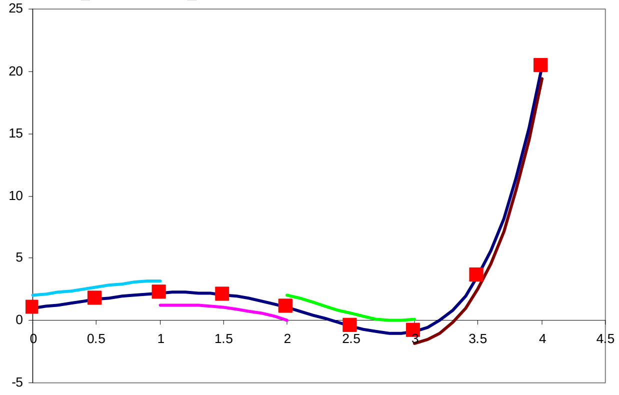
$\Rightarrow$Integrates a cubic exactly: $f^{(4)}(\xi) = 0$

Composite Simpson’s 1/3 Rule

- As with Trapezoidal rule, can use multiple applications of Simpson’s 1/3 rule.
- Need even number of intervals
  - An odd number of points are required.

Composite Simpson’s 1/3 Rule

- Example: 9 points, 4 intervals



Composite Simpson’s 1/3 Rule

- As in composite trapezoid, break integral up into *n*/2 sub-integrals:

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx$$

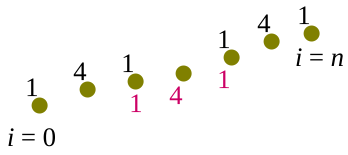
- Substitute Simpson’s 1/3 rule for each integral and collect terms.

$$I = (b-a)\frac{f(x_0)+4\sum_{i=1,3,5}^{n-1} f(x_i)+2\sum_{j=2,4,6}^{n-2} f(x_j)+f(x_n)}{3n}$$

n+1 data points, an odd number

Composite Simpson’s 1/3 Rule

- Odd coefficients receive a weight of 4, even receive a weight of 2.
- Doesn’t seem very fair, does it?



coefficients on numerator

Error Estimate

- The error can be estimated by:

$$E_a = \frac{nh^5}{180}\bar{f}^{(4)} = \frac{(b-a)h^4}{180}\bar{f}^{(4)} \qquad O(h^4)$$

- If n is doubled, $h \rightarrow h/2$ and $E_a \rightarrow E_a/16$

$\bar{f}^{(4)}$ is the average 4th derivative

Example

- Integrate $f(x) = e^{-x^2}$ from a = 0 to b = 2.
- Use Simpson's 1/3 rule:

$$h = \frac{b-a}{2} = 1 \qquad x_0 = a = 0 \qquad x_1 = \frac{a+b}{2} = 1 \qquad x_2 = b = 2$$

$$\begin{aligned} I = \int_0^2 e^{-x^2}dx &\approx \frac{1}{3}h\left[f(x_0) + 4f(x_1) + f(x_2)\right] \\ &= \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right] \\ &= \frac{1}{3}(e^0 + 4e^{-1} + e^{-4}) = 0.82994 \end{aligned}$$

Example

- Error estimate: $E_t = -\frac{h^5}{90} f^{(4)}(\xi)$
- Where $h = b - a$ and $a < \xi < b$
- Don't know ξ
  - use average value

$$E_t \approx E_a = -\frac{1^5}{90}\bar{f}^{(4)} = -\frac{1^5}{90}\frac{\left[f^{(4)}(x_0)+f^{(4)}(x_1)+f^{(4)}(x_2)\right]}{3}$$

Another Example

- Let's look at the polynomial again:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

  - From $a = 0$ to $b = 0.8$

$$h = \frac{b-a}{2} = 0.4 \qquad x_0 = a = 0 \qquad x_1 = \frac{a+b}{2} = 0.4 \qquad x_2 = b = 0.8$$

$$I = \int_0^2 f(x)dx \approx \frac{1}{3}h\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{(0.4)}{3}\left[f(0) + 4f(0.4) + f(0.8)\right]$$

$$= 1.36746667$$

Exact integral is 1.64053334

Error

- Actual Error: (using the known exact value)

  $E = 1.64053334 - 1.36746667 = 0.27306666$ 16%

- Estimate error: (if the exact value is not available)

  $$E_t = -\frac{h^5}{90} f^{(4)}(\xi)$$

- Where $a < \xi < b$.

Error

- Compute the fourth-derivative

  $$f^{(4)}(x) = -21600 + 48000x$$

$$E_t \approx E_a = -\frac{0.4^5}{90} f^{(4)}(x_1) = -\frac{0.4^5}{90} f^{(4)}(0.4) = 0.27306667$$

middle point

- Matches actual error pretty well.

Example Continued

- If we use 4 segments instead of 1: $h = \frac{b-a}{n} = 0.2$
  - **x** = [0.0 0.2 0.4 0.6 0.8]

$f(0) = 0.2$ $f(0.2) = 1.288$ $f(0.4) = 2.456$

$f(0.6) = 3.464$ $f(0.8) = 0.232$

$$I = (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$$= (0.8-0)\frac{f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + f(0.8)}{(3)(4)}$$

$$= 0.8\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12}$$

$$= 1.6234667$$

Exact integral is 1.64053334

Error

- Actual Error: (using the known exact value)

$$E = 1.64053334 - 1.6234667 = 0.01706667$$

1%

- Estimate error: (if the exact value is not available)

$$E_t \approx E_a = -\frac{0.2^5}{90} f^{(4)}(x_2) = -\frac{0.2^5}{90} f^{(4)}(\underset{\uparrow}{0.4}) = -0.0085$$

middle point

Error

- Actual is twice the estimated, why?
- Recall:

$$f^{(4)}(x) = -21600 + 48000x$$

$$\max_{x\in[0,0.8]}\left\{\left|f^{(4)}(x)\right|\right\} = \left|f^{(4)}(0)\right| = -21600$$

$$\left|f^{(4)}(0.4)\right| = 2400$$

Error

- Rather than estimate, we can bound the absolute value of the error:

$$\left|E_a\right| = \left|-\frac{0.2^5}{90} f^{(4)}(\xi)\right| \le \frac{0.2^5}{90}\left|f^{(4)}(0)\right| = 0.0768$$

- Five times the actual, but provides a safer error metric.

Simpon’s 1/3 Rule

- Simpson’s 1/3 rule uses a 2nd order polynomial
  - need 3 points or 2 intervals
  - This implies we need an even number of intervals.
- What if you don’t have an even number of intervals? Two choices:
  1. Use Simpson’s 1/3 on all the segments except the last (or first) one, and use trapezoidal rule on the one left.
     - Pitfall - larger error on the segment using trapezoid
  2. Use Simpson’s 3/8 rule.

Simpson’s 3/8 Rule

- Simpson’s 3/8 rule uses a third order polynomial
  - need 3 intervals (4 data points)

$$f(x) \approx p_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$I = \int_{x_0}^{x_3} f(x)dx \approx \int_{x_0}^{x_3} p_3(x)dx$$

Simpson's 3/8 Rule

- Determine *a*'s with Lagrange polynomial
- For evenly spaced points

$$I = \frac{3}{8}h\left[f\left(x_0\right)+3\left(x_1\right)+3f\left(x_2\right)+f\left(x_3\right)\right]$$

$$h = \frac{b-a}{3}$$

Error

- Same order as 1/3 Rule.
  - More function evaluations.
  - Interval width, *h*, is smaller.

$$E_t = -\frac{3}{80}h^5 f^{(4)}(\xi) \qquad O(h^4)$$

- Integrates a cubic exactly:

$$\Rightarrow \qquad f^{(4)}(\xi) = 0$$

Comparison

- Simpson’s 1/3 rule and Simpson’s 3/8 rule have the same order of error
  - $O(h^4)$
  - trapezoidal rule has an error of $O(h^2)$
- Simpson’s 1/3 rule requires **even number** of segments.
- Simpson’s 3/8 rule requires multiples of **three segments.**
- Both Simpson’s methods require **evenly spaced** data points

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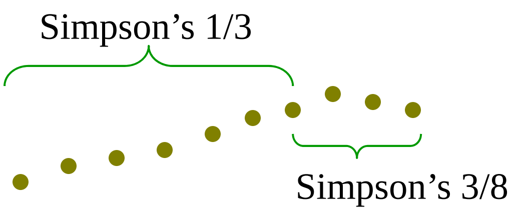
Mixing Techniques

- n = 10 points ⇒ 9 intervals
  - First 6 intervals - Simpson’s 1/3
  - Last 3 intervals - Simpson’s 3/8



Newton-Cotes Formulas

- We can examine even higher-order polynomials.
  - Simpson’s 1/3 - 2nd order Lagrange (3 pts)
  - Simpson’s 3/8 - 3rd order Lagrange (4 pts)
- Usually do not go higher.
- Use multiple segments.
  - But only where needed.

Adaptive Simpson’s Scheme

- Recall Simpson’s 1/3 Rule:

$$\boxed{I \approx \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]}$$

- Where initially, we have $a=x_0$ and $b=x_2$.
- Subdividing the integral into two:

$$I \approx \frac{h}{6}\left[f(a)+4f(x_1)+2f(x_2)+4f(x_3)+f(b)\right]$$

Adaptive Simpson's Scheme

- We want to keep subdividing, until we reach a desired error tolerance, ε.
- Mathematically:

$$\left|\int_a^b f(x)\,dx - \left[\frac{h}{3}\left[f(a)+4f(x_1)+f(b)\right]\right]\right| \le \varepsilon$$

$$\left|\int_a^b f(x)\,dx - \left[\frac{h}{6}\left[f(a)+4f(x_1)+2f(x_2)+4f(x_3)+f(b)\right]\right]\right| \le \varepsilon$$

Adaptive Simpson's Scheme

- This will be satisfied if:

$$\left|\int_a^c f(x)\,dx - \left[\frac{h}{6}\left[f(a)+4f(x_1)+f(x_2)\right]\right]\right| \le \frac{\varepsilon}{2},\quad \textit{and}$$

$$\left|\int_c^b f(x)\,dx - \left[\frac{h}{6}\left[f(x_2)+4f(x_3)+f(b)\right]\right]\right| \le \frac{\varepsilon}{2},\quad \textit{where}$$

$$c = x_2 = \frac{a+b}{2}$$

- The left and the right are within one-half of the error.

Adaptive Simpson’s Scheme

- Okay, now we have two separate intervals to integrate.
- What if one can be solved accurately with an $h=10^{-3}$, but the other requires many, many more intervals, $h=10^{-6}$?

Adaptive Simpson’s Scheme

- Adaptive Simpson’s method provides a divide and conquer scheme until the appropriate error is satisfied everywhere.
- Very popular method in practice.
- Problem:
  - We do not know the exact value, and hence do not know the error.

Adaptive Simpson's Scheme

- How do we know whether to continue to subdivide or terminate?

$$I \equiv \int_a^b f(x)\,dx = S(a,b) + E(a,b),\ \ \textit{where}$$

$$S(a,b) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],\ \ \textit{and}$$

$$E(a,b) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}$$

Adaptive Simpson's Scheme

- The first iteration can then be defined as:

$$I = S^{(1)} + E^{(1)},\ \ \textit{where}$$

$$S^{(1)} = S(a,b),\ E^{(1)} = E(a,b)$$

- Subsequent subdivision can be defined as:

$$S^{(2)} = S(a,c) + S(c,b)$$

Adaptive Simpson's Scheme

- Now, since

$$E^{(2)} = E(a,c) + E(c,b)$$

- We can solve for $E^{(2)}$ in terms of $E^{(1)}$.

$$E^{(2)} = -\frac{1}{90}\left(\frac{h/2}{2}\right)^5 f^{(4)} - \frac{1}{90}\left(\frac{h/2}{2}\right)^5 f^{(4)}$$

$$= \left(\frac{1}{2^4}\right) - \frac{1}{90}\left(\frac{h}{2}\right)^5 f^{(4)} = \frac{1}{16}E^{(1)}$$

Adaptive Simpson's Scheme

- Finally, using the identity:

$$I = S^{(1)} + E^{(1)} = S^{(2)} + E^{(2)}$$

- We have:

$$S^{(2)} - S^{(1)} = E^{(1)} - E^{(2)} = 15E^{(2)}$$

- Plugging into our definition:

$$I = S^{(2)} + E^{(2)} = S^{(2)} + \frac{1}{15}\left(S^{(2)} - S^{(1)}\right)$$

Adaptive Simpson's Scheme

- Our error criteria is thus:

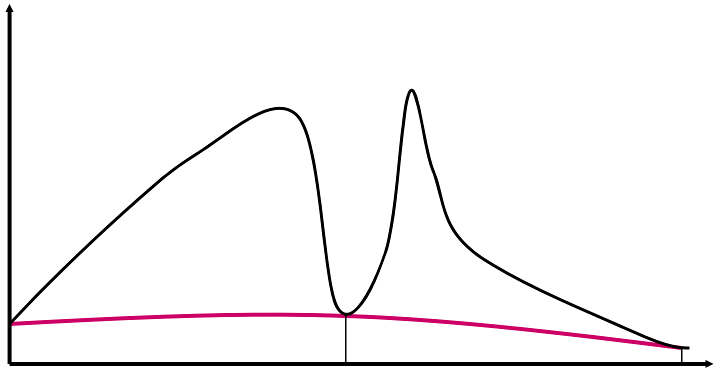
$$\left| I - S^{(2)} \right| = \left| \frac{1}{15}\left(S^{(2)} - S^{(1)} \right) \right| \le \varepsilon$$

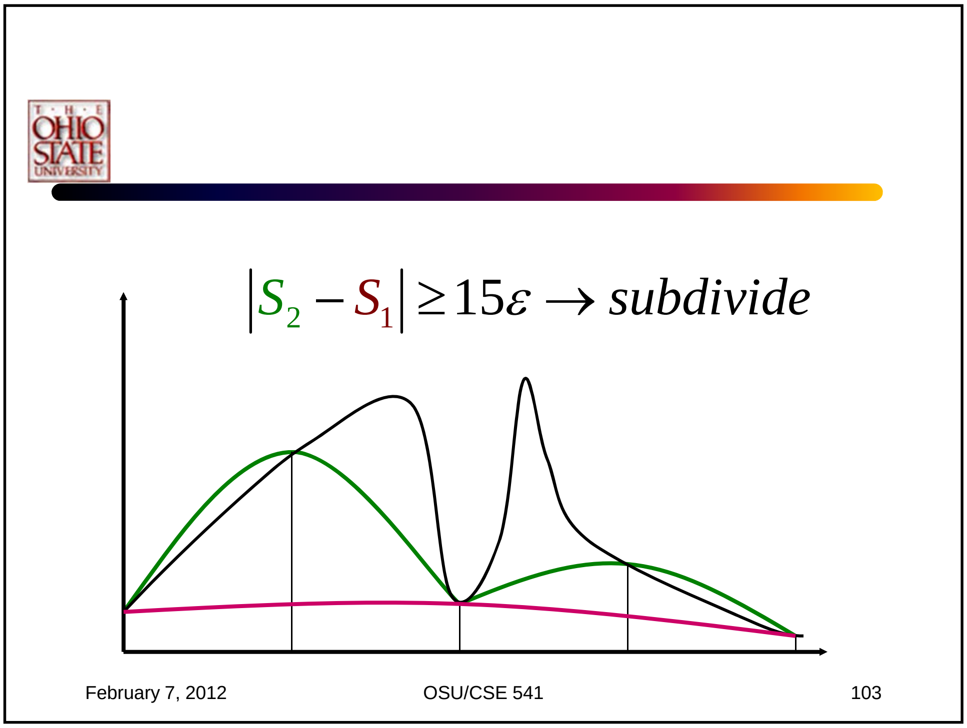
- Simplifying leads to the termination formula:

$$\boxed{\left| \left(S^{(2)} - S^{(1)} \right) \right| \le 15\varepsilon}$$

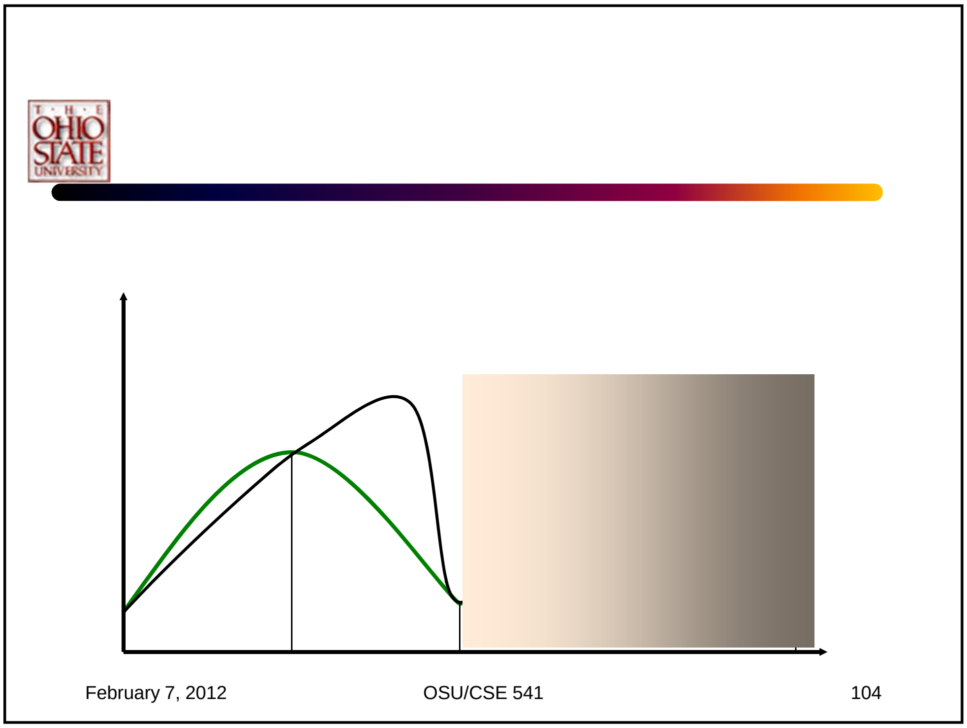
Adaptive Simpson's Scheme

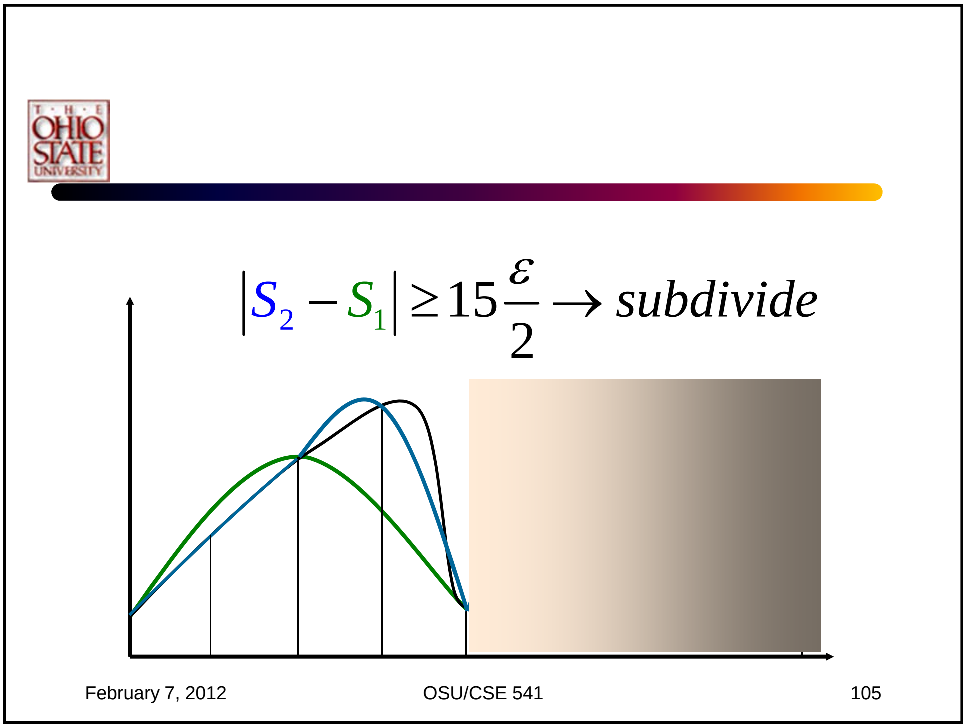
- What happens graphically:





$$\left|S_2 - S_1\right| \geq 15\varepsilon \rightarrow \mathit{subdivide}$$





$$\left|S_2 - S_1\right| \geq 15\frac{\varepsilon}{2} \rightarrow subdivide$$

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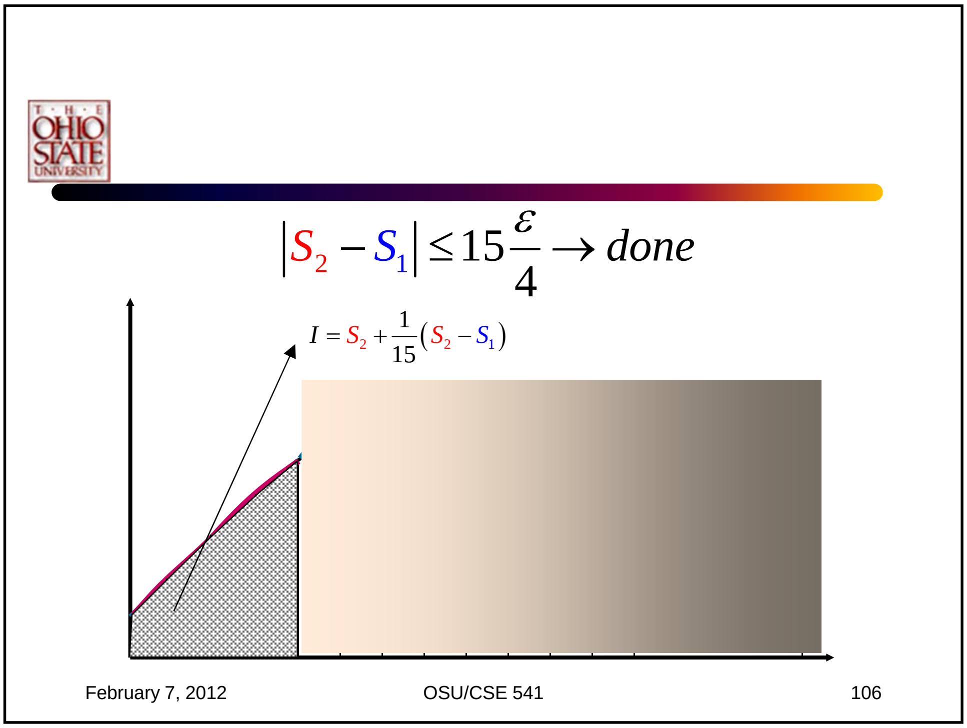

$$\left|S_2 - S_1\right| \leq 15\frac{\varepsilon}{4} \rightarrow done$$

$I = S_2 + \frac{1}{15}\left(S_2 - S_1\right)$

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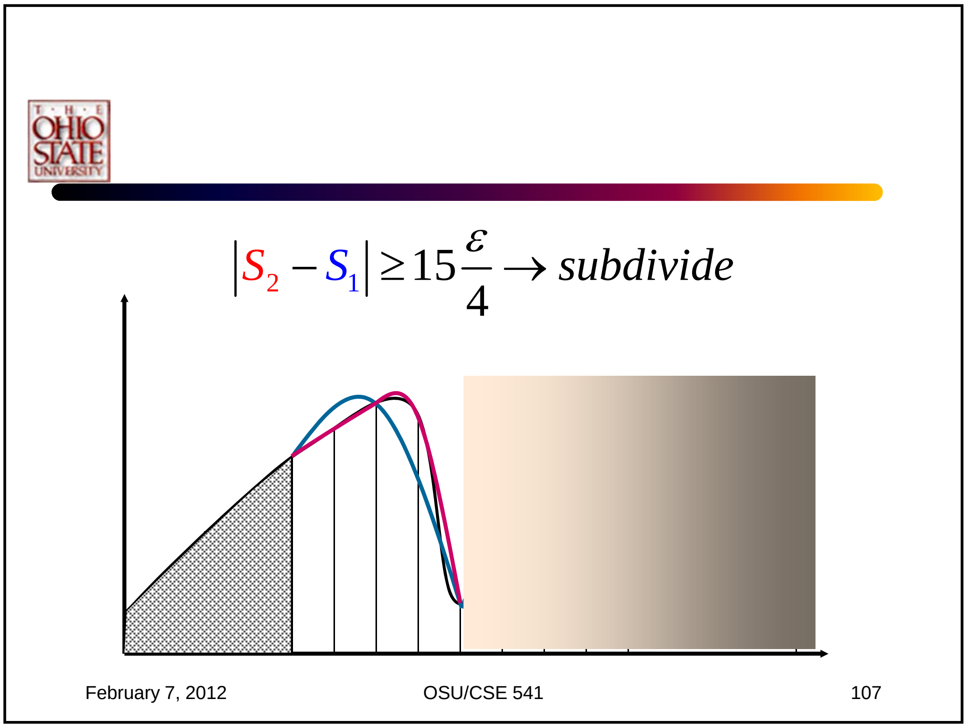
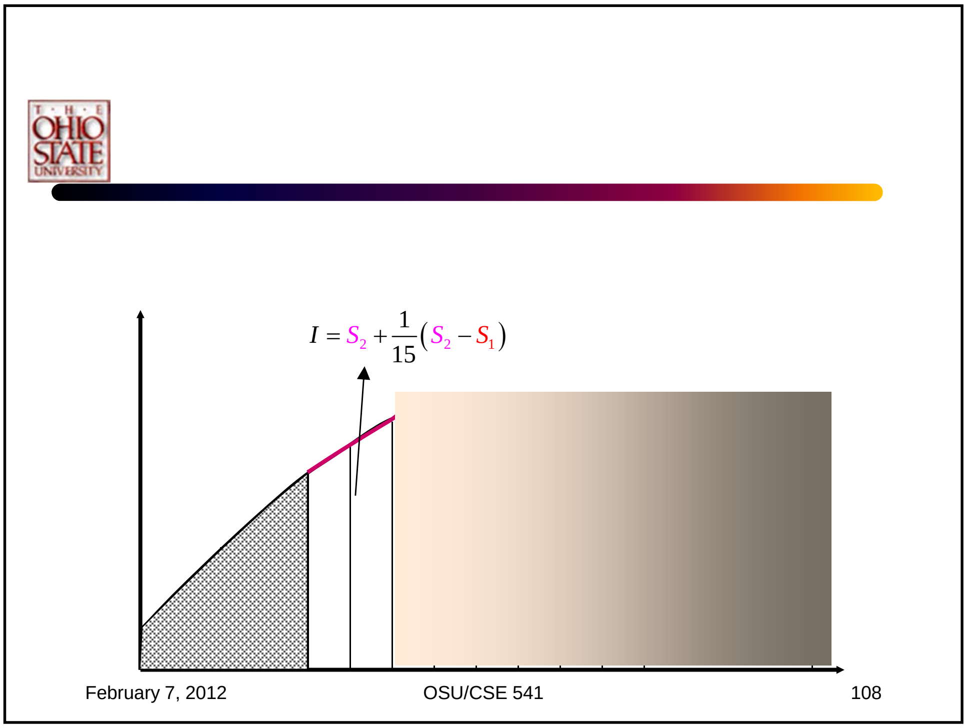
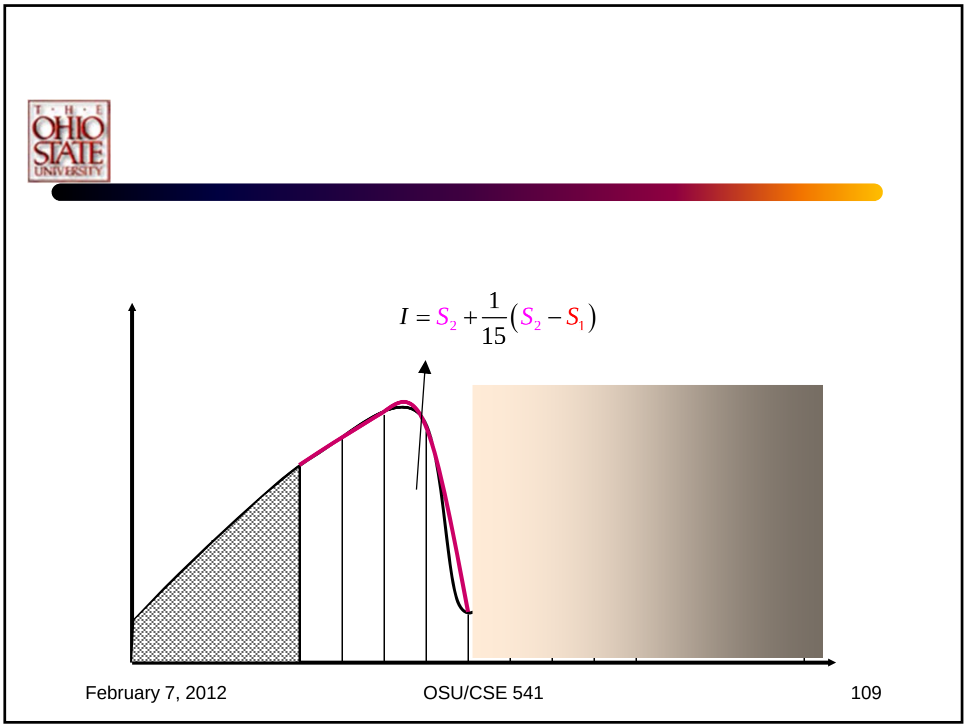


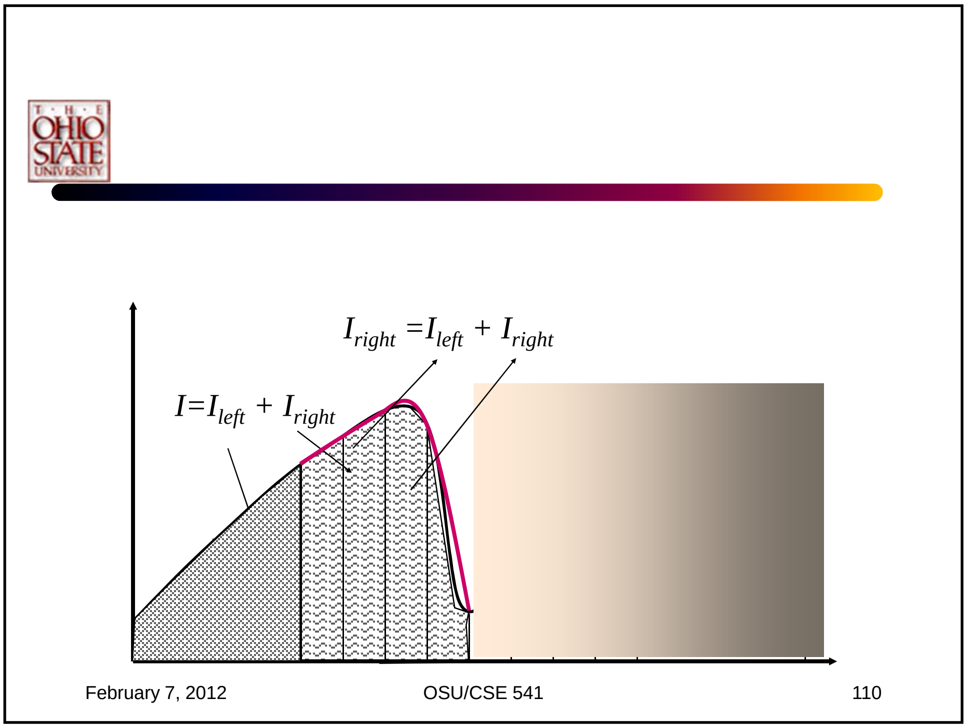
$$\left|S_2 - S_1\right| \geq 15\frac{\varepsilon}{4} \rightarrow subdivide$$



$$I = S_2 + \frac{1}{15}\left(S_2 - S_1\right)$$

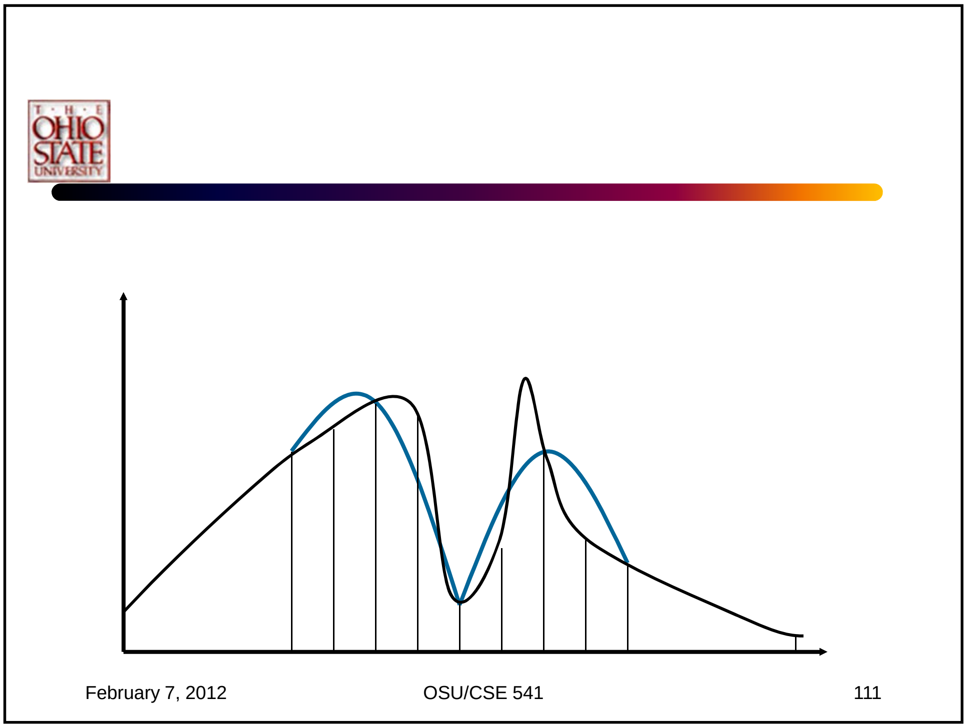


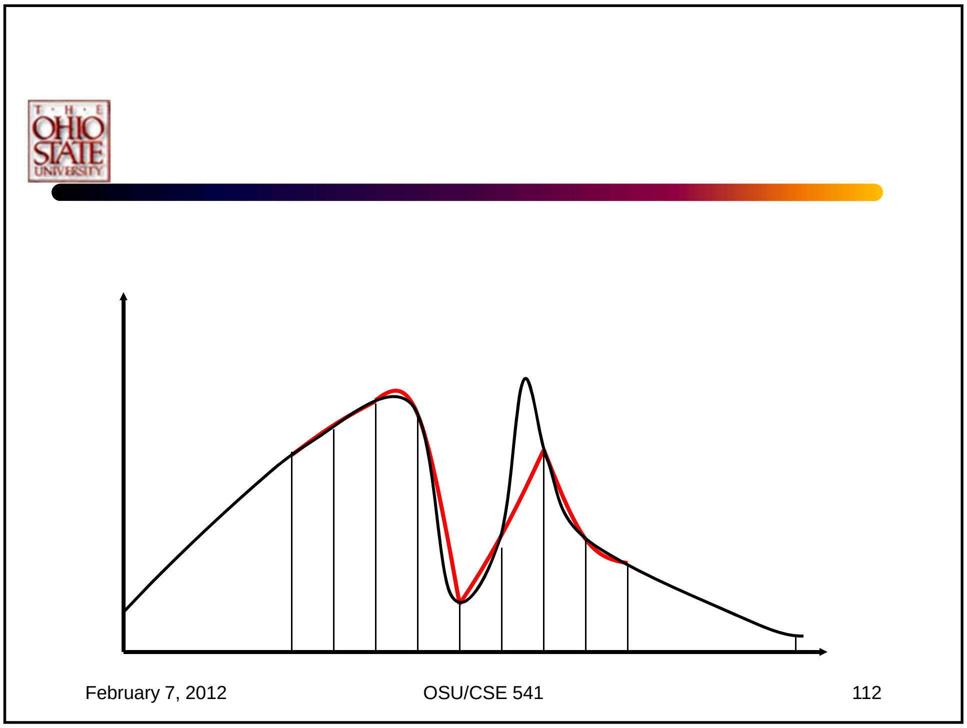
$$I = S_2 + \frac{1}{15}\left(S_2 - S_1\right)$$



$$I_{right} = I_{left} + I_{right}$$

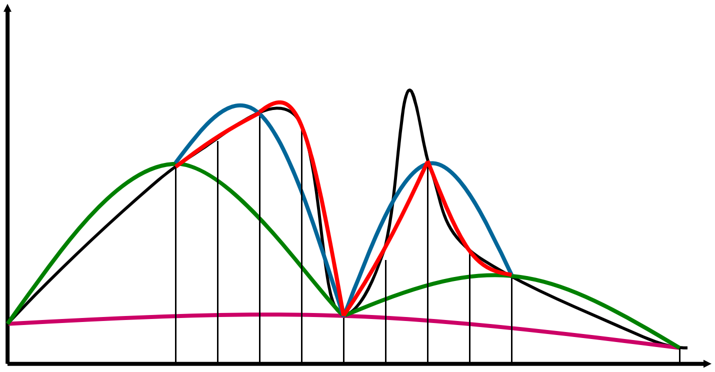
$$I = I_{left} + I_{right}$$





Adaptive Simpson's Scheme

- We gradually capture the difficult spots.



Adaptive Simpson's Code

- Simple Recursive Program

```
static const int m_nMaximum_Divisions = 1000;
Real IntegrationSimpson( const Real (*f) (Real x), const Real start, const Real end, const Real
                                                    error_tolerance, int &level )
{
        level += 1;
        Real h = (end – start);
        Real midpoint = (start + end) / 2.0;
        Real f_start = f(start);
        Real f_end = f(end);
        Real f_mid = f(midpoint );
        oneLevel = h*( f_start + 4.0*f_mid + f_end) / 6.0;
        Real leftMidpoint = (start+ midpoint ) / 2.0;
        Real rightMidpoint = (end+ midpoint ) / 2.0
        Real f_midLeft = f(leftMidpoint );
        Real f_midRight = f(rightMidpoint );
        twoLevel = h*(f_start + 4.0* f_midLeft + 2.0* f_mid + 4.0* f_midRight + f_end) / 12.0;
        if( level >= m_nMax_Divisions ) // Terminate the process, converging too slow
                return twoLevel;
```

Adaptive Simpson's Code

```
        if( absf( twoLevel – oneLevel) < 15.0*error_tolerance) // Desired solution reached
                return twoLevel + (twoLevel-oneLevel) / 15.0;
        //
        // Otherwise, split the interval in two and recursively evaluate each half.
        //
        leftIntegral = IntegrationSimpson( f, start, midpoint , error_tolerance/2.0, level );
        rightIntegral = IntegrationSimpson( f, midpoint , end, error_tolerance/2.0, level );
        return leftIntegral + rightIntegral;
}
```

Guassian Quadrature

- Idea is that if we evaluate the function at certain points, and sum with certain weights, we will get a more accurate integral
- Evaluation points and weights are pre-computed and tabulated
- Basic form: $$I = \int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

c_i : weighting factors
x_i : sampling points selected optimally New!!

Guassian Quadrature

- Note that the interval is between –1 and 1
- For other intervals, a change of variables is used to transfer the problem so that it utilizes the interval [-1, 1]
- This is a linear transform, such that for $t \in [a,b]$:

$$\int_a^b f(t)dt$$

- We have for $x \in [-1,1]$:

$$t = \frac{(b-a)x+b+a}{2} \Longrightarrow x = \frac{2t-b-a}{b-a}$$

Guassian Quadrature

- As $t = a \Rightarrow x = -1$
- As $t = b \Rightarrow x = 1$

$$dt = \frac{(b-a)}{2}dx$$

$$f(t) = f\left(\frac{(b-a)x+b+a}{2}\right)$$

$$\int_a^b f(t)dt = \frac{(b-a)}{2}\int_{-1}^{1} f\left(\frac{(b-a)x+b+a}{2}\right)dx$$

Guassian Quadrature

- Basic form of Gaussian quadrature:

$$I = \int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

- For n=2, we have:

$$I \approx c_1 f\left(x_1\right) + c_2 f\left(x_2\right)$$

- This leads to 4 unknowns: c_1, c_2, x_1, and x_2
  - two unknown weights (c_1, c_2)
  - two unknown sampling points (x_1, x_2)

Guassian Quadrature

- What we need now, are four known values for the equation.
- If we had these, we could then attempt to solve for the four unknowns.
- Let’s make it work for polynomials!!!

Guassian Quadrature

- In particular, let’s look at these simple polynomials:
  - Constant
    - *f(x)=1*
  - Linear
    - *f(x)=x*
  - Quadratic
    - $f(x)=x^2$
  - Cubic
    - $f(x)=x^3$

Guassian Quadrature

- Recalling the formula: $I \approx c_1 f(x_1) + c_2 f(x_2)$
  - Constant
    - *f(x)=1*

$$\int_{-1}^{1} 1dx = 2 \quad = c_1 f(x_1) + c_2 f(x_2) = c_1 + c_2$$

  - Linear
    - *f(x)=x*

$$\int_{-1}^{1} xdx = 0 \quad = c_1 f(x_1) + c_2 f(x_2) = c_1 x_1 + c_2 x_2$$

  - Quadratic
    - $f(x)=x^2$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} = c_1 f(x_1) + c_2 f(x_2) = c_1 x_1^{\,2} + c_2 x_2^{\,2}$$

  - Cubic
    - $f(x)=x^3$

$$\int_{-1}^{1} x^3 dx = 0 \quad = c_1 f(x_1) + c_2 f(x_2) = c_1 x_1^{\,3} + c_2 x_2^{\,3}$$

Guassian Quadrature

- We can now solve for our unknowns:
  - *Note, this is not an easy problem and will not be covered in this class.*

$$c_1 = c_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}} = -0.577$$

$$x_2 = \frac{1}{\sqrt{3}} = 0.577$$

Guassian Quadrature

- This yields the two point **Gauss-Legendre** formula

$$I \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Guassian Quadrature

- This is **<u>exact</u>** for **all** polynomials up to and including degree 3 (cubics).

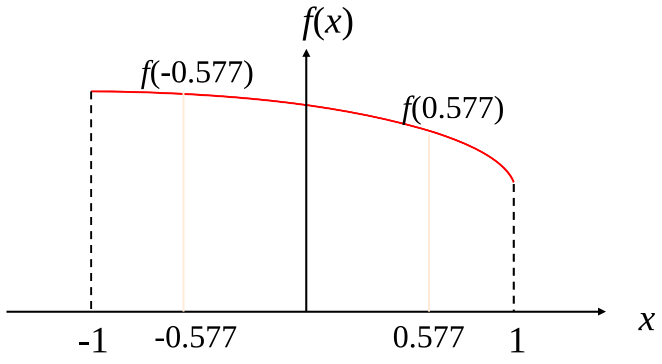
$$\int_{-1}^{1}\left(ax^3+bx^2+cx+d\right)dx = a\int_{-1}^{1}x^3dx+b\int_{-1}^{1}x^2dx+c\int_{-1}^{1}xdx+d\int_{-1}^{1}dx$$

$$=a\left(\left(\frac{-1}{\sqrt{3}}\right)^3+\left(\frac{1}{\sqrt{3}}\right)^3\right)+b\left(\left(\frac{-1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2\right)+c\left(\left(\frac{-1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}\right)\right)+d\left(1+1\right)$$

$$=\left(ax^3+bx^2+cx+d\right)\Big|_{\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}$$

Guassian Quadrature

$$\int_{-1}^{1} f(x)dx \approx f(-0.577)+f(0.577)$$



Example

- Integrate *f(x)* from *a* = 0 to *b* = 0.8

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

- Transform from [0, 0.8] to [-1, 1]

$$\int_a^b f(t)dt = \frac{(b-a)}{2}\int_{-1}^{1} f\left(\frac{(b-a)x + b + a}{2}\right)dx$$

$$I = \int_0^{0.8} f(x)dx = \frac{(0.8-0)}{2}\int_{-1}^{1} f\left(\frac{(0.8-0)t + 0.8 + 0}{2}\right)dt$$

$$= 0.4\int_{-1}^{1} f(0.4t + 0.4)dt$$

Example

- Solving

$$I = 0.4\int_{-1}^{1} f(0.4t + 0.4)dt$$

$$= 0.4\int_{-1}^{1}\begin{bmatrix} 0.2 + 25(0.4t - 0.4) - 200(0.4t - 0.4)^2 \\ + 675(0.4t - 0.4)^3 - 900(0.4t - 0.4)^4 + 400(0.4t - 0.4)^5 \end{bmatrix}dt$$

- And substituting for the 2-point formula:

$$I = 0.4\int_{-1}^{1} f(t)dt \qquad t = \pm 1/\sqrt{3}$$

$$I \approx 0.51674055 + 1.30583723 = 1.82257778$$

Exact integral is **1.64053334**

Higher-order Gaussian Quadrature

- Recall the basic form:

$$I = \int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

- Let’s look at n=3.

$$I \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

- We now have 6 unknowns: c_1, c_2, c_3,x_1, x_2, and x_3
  - three unknown weights (c_1, c_2 , c_3)
  - three unknown sampling points (x_1, x_2 , x_3)

Use 6 equations - constant, linear, quadratic, cubic, 4th order and 5th order to find those unknowns

$$\int_{-1}^{1} 1dx = 2 \quad = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 + c_2 + c_3$$

$$\int_{-1}^{1} xdx = 0 \quad = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^{\ 2} + c_2 x_2^{\ 2} + c_3 x_3^{\ 2}$$

$$\int_{-1}^{1} x^3 dx = 0 \ = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^{\ 3} + c_2 x_2^{\ 3} + c_3 x_3^{\ 3}$$

$$\int_{-1}^{1} x^4 dx = \frac{2}{5} \ = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^{\ 4} + c_2 x_2^{\ 4} + c_3 x_3^{\ 4}$$

$$\int_{-1}^{1} x^5 dx = 0 \ = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^{\ 5} + c_2 x_2^{\ 5} + c_3 x_3^{\ 5}$$

Higher-order Gaussian Quadrature

- Can solve these equations (or have some one smarter than us, like Guass solve them).

$$c_1 = 5/9 \qquad c_2 = 8/9 \qquad c_3 = 5/9$$

$$x_1 = -\sqrt{3/5} = -0.77459669 \qquad x_2 = 0 \qquad x_2 = \sqrt{3/5} = 0.77459669$$

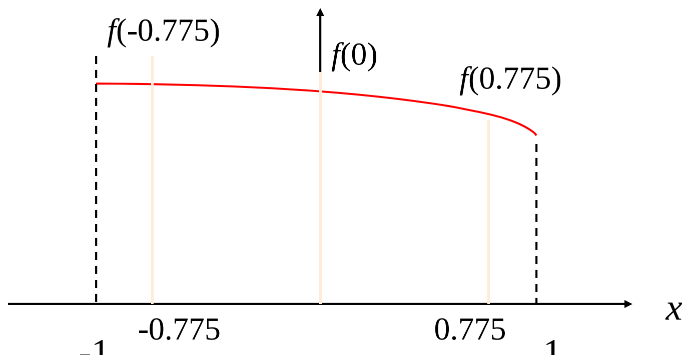
- Produces the three point Gauss-Legendre formula

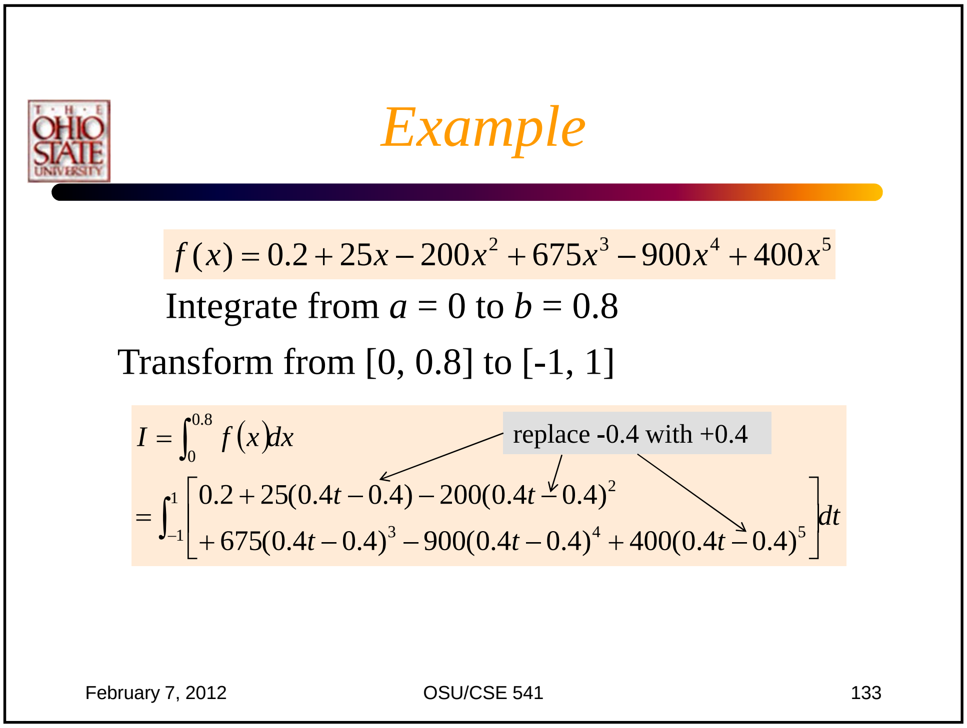
$$I \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

  - Exact for polynomials up to and including degree 5 (because using 5th degree polynomial)

Higher-order Gaussian Quadrature

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f(-0.775) + \frac{8}{9} f(0.0) + \frac{5}{9} f(0.775)$$





Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Integrate from *a* = 0 to *b* = 0.8

Transform from [0, 0.8] to [-1, 1]

$$I = \int_0^{0.8} f(x)dx$$

replace -0.4 with +0.4

$$= \int_{-1}^{1} \begin{bmatrix} 0.2 + 25(0.4t - 0.4) - 200(0.4t - 0.4)^2 \\ + 675(0.4t - 0.4)^3 - 900(0.4t - 0.4)^4 + 400(0.4t - 0.4)^5 \end{bmatrix} dt$$

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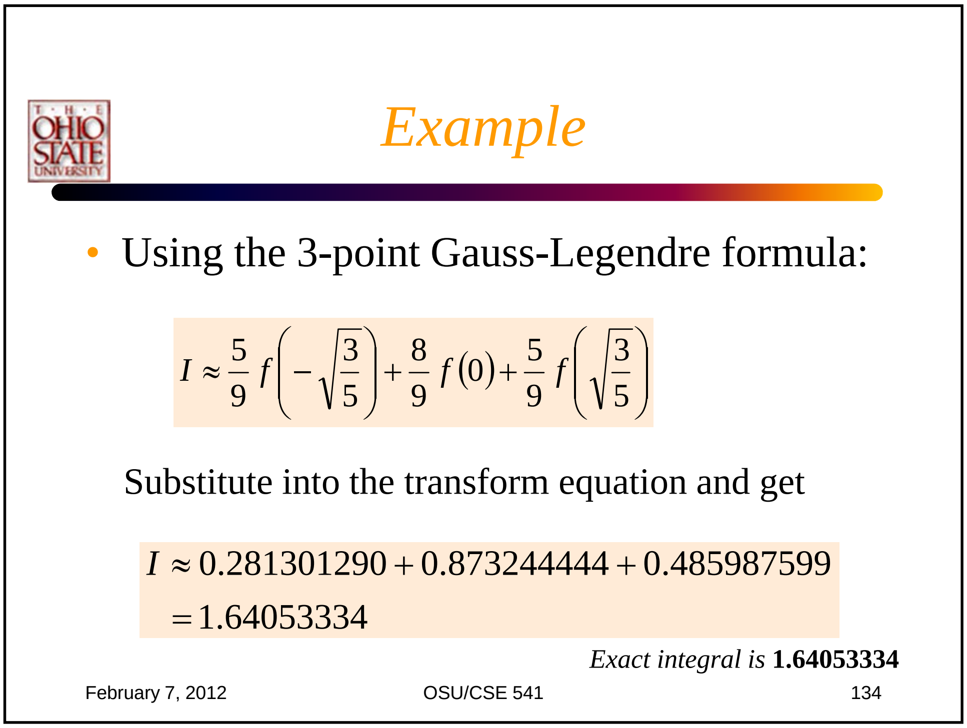

Example

- Using the 3-point Gauss-Legendre formula:

$$I \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

Substitute into the transform equation and get

$$I \approx 0.281301290 + 0.873244444 + 0.485987599$$
$$= 1.64053334$$

Exact integral is **1.64053334**

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Gaussian Quadrature

Can develop higher order Gauss-Legendre forms using

$$I \approx c_1 f\left(x_1\right)+c_2 f\left(x_2\right)+\ldots+c_n f\left(x_n\right)$$

Values for *c*’s and *x*’s are tabulated

Use the same transformation to map interval onto [-1, 1]

$$I=\int_{-1}^{1} f(x)dx \approx c_1 f\left(x_1\right)+c_2 f\left(x_2\right)+\ldots+c_n f\left(x_n\right)$$

n	2	3	4	5	6
c_i	1.0	0.5555555556	0.3478548451	0.2369268850	0.1713245
	1.0	0.8888888889	0.6521451549	0.4786286705	0.3607616
		0.5555555556	0.6521451549	0.5688888889	0.4679139
			0.3478548451	0.4786286705	0.4679139
				0.2369268850	0.3607616
					0.1713245
x_i	-0.5773502692	-0.7745966692	-0.8611363116	-0.9061798459	-0.932469514
	0.5773502692	0.0000000000	-0.3399810436	-0.5384693101	-0.661209386
		0.7745966692	0.3399810436	0.0000000000	-0.238619186
			0.8611363116	0.5384693101	0.238619186
				0.9061798459	0.661209386
					0.932469514

Gaussian Quadrature

- Requires function evaluations at non-uniformly spaced points within the integration interval
  - not appropriate for cases where the function is unknown
  - not suited for dealing with tabulated data that appear in many engineering problems
- If the function is known, its efficiency can be a decided advantage

Gaussian Quadrature

- Problems:
  - If we add more data points, like doubling the number of sample points.