Dimension Detection by Local Homology

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Department of Computer Science and Engineering
The Ohio State University

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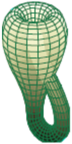
Joint work with Fengtao Fan and Yusu Wang

Introduction

⋮

(0.80 0.01 0.60 0.00)
(0.98 0.03 0.17 0.00)
(0.58 0.05 0.81 0.03)
(0.26 0.05 0.96 0.09)

⋮



Points in $\mathbb{R}^4$

Points in $\mathbb{R}^{4096}$

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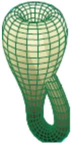

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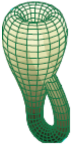

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Topological Method

Simplicial Complex

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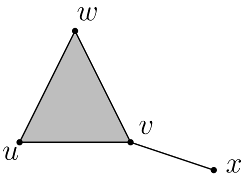

$\mathcal{K}$

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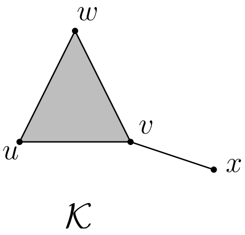

$$\mathcal{K} = \{ \{u\}, \{v\}, \{w\}, \{x\}, \{uv\}, \{uw\}, \{vw\}, \{vx\}, \{uvw\} \}$$

Rips Complex

- A *Rips complex* $\mathcal{R}^r(Q)$ on a finite metric space (Q, d) consists of all simplices $\{q_0, q_1, \ldots, q_k\}$ $(q_i \in Q)$
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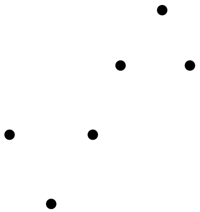

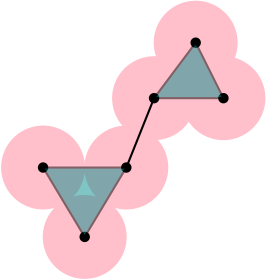

Homology, Relative Homology and Local Homology under $\mathbb{Z}_2$

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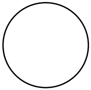

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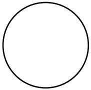

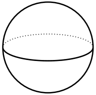

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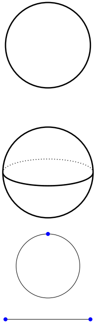

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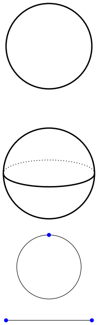

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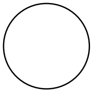

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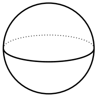

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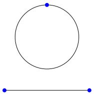
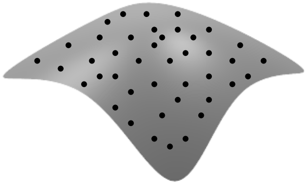


- Local homology at $p \in X$: $\mathsf{H}(X, X - p)$
  - ▷ $\mathcal{M}$ m-dimensional manifold, $p \in \mathcal{M}$

$$\mathsf{H}_n(\mathcal{M}, \mathcal{M} - p) = \tilde{\mathsf{H}}_n(\mathbb{S}^m) = \begin{cases} \mathbb{Z}_2 & n = m \\ 0 & n \neq m \end{cases}$$

Goal

- Points P from $\mathcal{M}$



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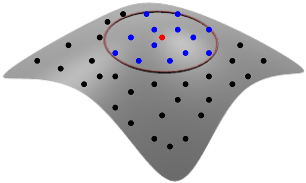

- Infer the local homology $\mathrm{H}(\mathcal{M}, \mathcal{M} - z)$ by using local points around z;
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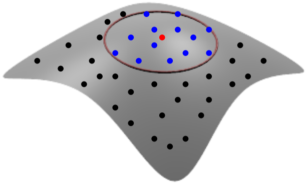
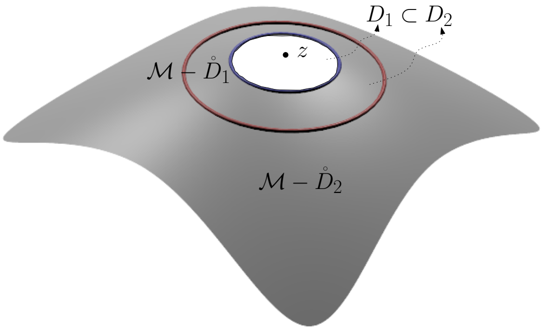


- Infer the local homology $\mathrm{H}(\mathcal{M}, \mathcal{M} - z)$ by using local points around z;
  - Build Rips complexes on local points;
  - Persistence image of Rips complexes;

Smooth Manifold



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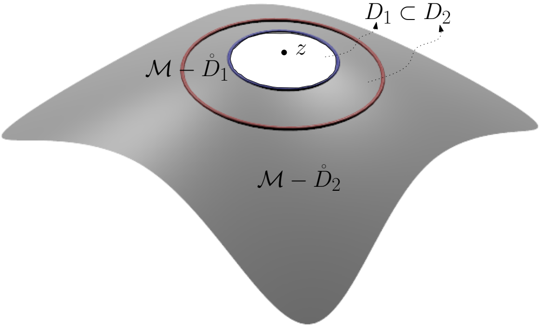

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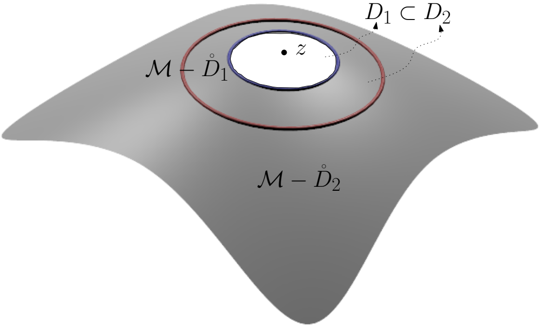

- $\mathcal{M}$ smooth manifold;
- $D_i \subset \mathcal{M}$ topological balls $(i = 1, 2)$;
- $z \in D_1 \subset D_2$;

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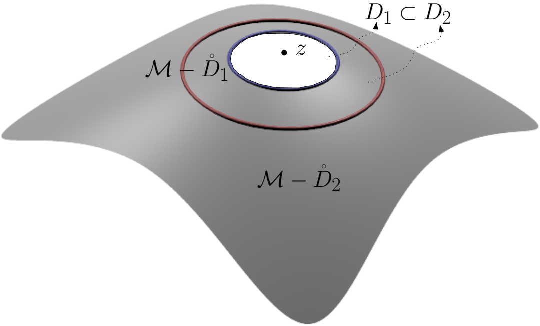

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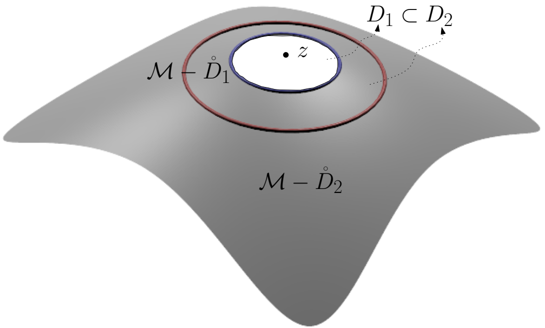

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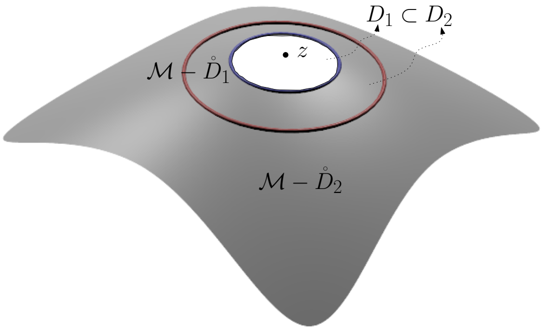

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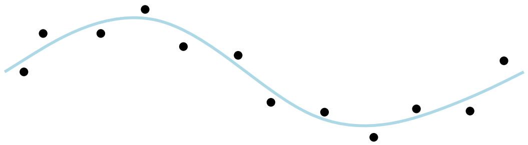
Union of Balls



- Sample P of $\mathcal{M}$;

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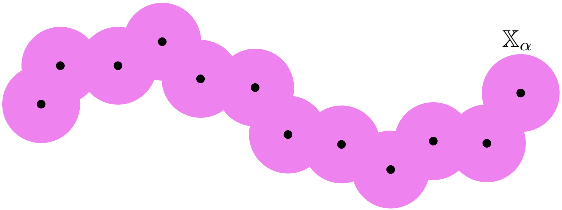
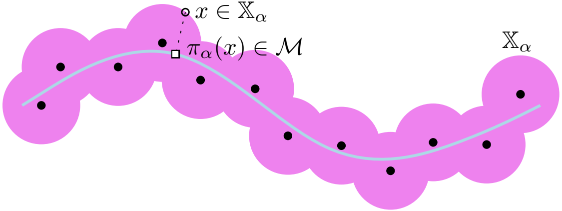


- Sample P of $\mathcal{M}$;
- $\mathbb{X}_\alpha = \cup_{q \in P} B_\alpha(q)$;

Union of Balls



- $\pi_\alpha : \mathbb{X}_\alpha \to \mathcal{M}$;

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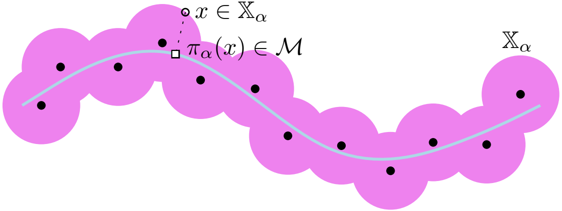

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- π_α deformation retraction;

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Local Homology by Union of Balls

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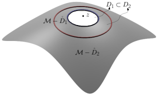

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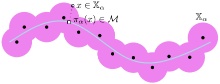

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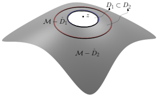

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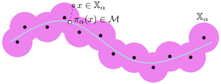

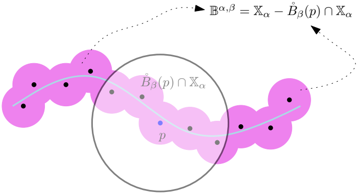

$$\mathrm{Im}(\mathsf{H}(\mathbb{X}_\alpha, \mathbb{B}^{\alpha,\lambda+3\delta}) \to \mathsf{H}(\mathbb{X}_{\alpha'}, \mathbb{B}^{\alpha',\lambda'+\delta'})) \cong \mathsf{H}(\mathcal{M}, \mathcal{M} - \bar{p})$$

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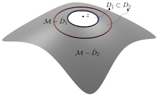

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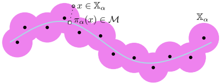
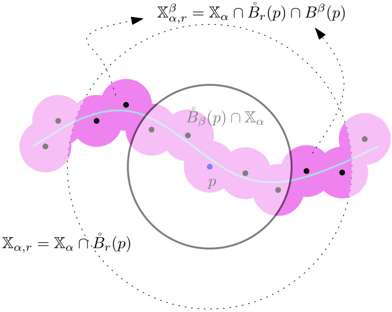
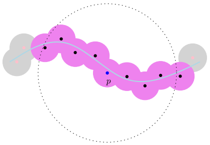




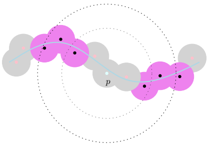
$$\mathrm{Im}(\mathsf{H}(\mathbb{X}_{\alpha,r}, \mathbb{X}^{\lambda+3\delta}_{\alpha,r}) \rightarrow \mathsf{H}(\mathbb{X}_{\alpha',r}, \mathbb{X}^{\lambda'+\delta'}_{\alpha',r})) \cong \mathsf{H}(\mathcal{M}, \mathcal{M} - \bar{p})$$

Local Homology by Rips Complexes

$P_{\alpha,r} = \{p_i \in P : B_\alpha(p_i) \cap \mathring{B}_r(p) \neq \emptyset\}$



$P^{\beta}_{\alpha,r} = \{p_i \in P_{\alpha,r} : B_\alpha(p_i) \cap B^{\beta}(p) \neq \emptyset \}$



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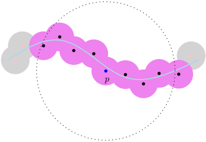

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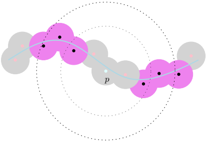
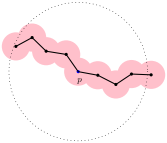


$$\mathcal{R}^{2\alpha}(P_{\alpha,r})$$



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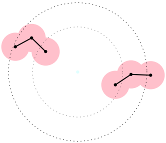

Local Homology by Rips Complexes

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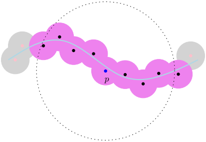

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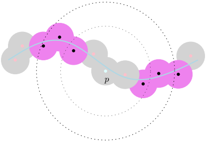

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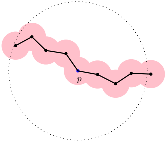

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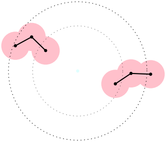


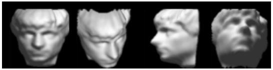
Theorem

For appropriate parameters, the inclusion $j_\alpha : (\mathcal{R}^{2\alpha}(P_{\alpha,r}), \mathcal{R}^{2\alpha}(P^{\eta_2}_{\alpha,r})) \hookrightarrow (\mathcal{R}^{6\alpha}(P_{3\alpha,r}), \mathcal{R}^{6\alpha}(P^{\eta_1}_{3\alpha,r}))$ *satisfies*

$$\mathrm{Im}(j_{\alpha *}) \cong \mathsf{H}(\mathcal{M}, \mathcal{M} - \bar{p}).$$

Experimental Results

- Image data:



(a) Rotating **Head**

(b) Handwritten **D1**

(c) Handwritten **D0**

- Comparison results:

	Shift	Head	D1	D0
Ours	2	3	4	3
SLIVER	3	4	3	2
MLE	4.27	4.31	11.47	14.86
MA	3.35	4.47	10.77	13.93
PN	3.62	3.98	6.22	8.86
LPCA	3	3	5	8.86
ISOMAP	2	3	5	[3, 6]

SLIVER: the method via slivers;
MLE: the maximum likelihood estimation;
MA: the manifold adaptive method;
PN: the packing number method;
LPCA: the local PCA;
ISOMAP: the isomap method;

THANKS!