Supplementary Material for SIGGRAPH 2013 paper: *An Efficient Computation of Handle and Tunnel Loops via Reeb Graphs*

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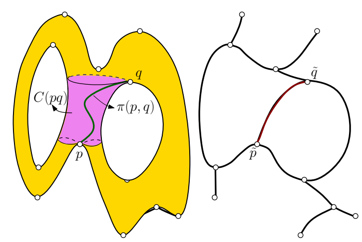
1 Loops and Their Perturbation

1.1 Mapping from the Reeb graph to the surface (Step 3)

A Reeb graph loop c_e consists of a set of Reeb graph arcs. For each arc $Arc(pq)$ with endpoints p and q being non-regular nodes for Rb_{M}, we compute a path $\pi(p,q) \subset \mathrm{M}$ that will be mapped to $Arc(pq)$ under the surjection map $\Phi : \mathrm{M} \to \mathrm{Rb}_{\mathrm{M}}$. The concatenation of these paths will form a loop γ_i in the surface M corresponding to c_e; that is, $\Phi(\gamma_i) = c_e$.



Specifically, note that the pre-image of $Arc(pq)$ is the evolution of a single contour till it either splits, merges with another one, or is destroyed. We call this the *topological component* $C(pq)$ corresponding to $Arc(pq)$. See the right figure for an illustration, where the shaded region in M is $C(pq)$. We can obtain $C(pq)$ by computing the contours passing through p and q and cutting and refining any triangle intersecting these contours. We then compute any path in $C(pq)$ from p to q by a depth first search on the edges in $C(pq)$, and use this path as $\pi(p,q)$. Note that some edges in $\pi(p,q)$ may be the refined edges along the contours. For each Reeb graph arc $Arc(pq)$, the computation of $C(pq)$ as well as of a pre-image path $\pi(p,q)$ takes $O(n)$ time. Since there are $O(n)$ number or arcs in the Reeb graph, it takes $O(n^2)$ time to compute a path in the pre-image for every Reeb graph arc. Given a Reeb graph loop c_e, we compute a loop γ_i in M by simply assembling the pre-image paths for all arcs in c_e. This step takes $O(ng)$ time for all g Reeb graph loops.

We remark that the loops in γ_is may contain edges produced by locally cutting a triangle with a contour. However, later in Section 3.2, we map these edges to the original mesh edges in M before we perform tightening of handle / tunnel loops. Hence, the final loops output by our algorithm consist of only edges from input mesh.

1.2 Properties of Loops $\{\gamma_i\} \cup \{\overline{\gamma}_i\}$ (Step 3)

Here we prove Property (L0), namely, the set of loops $\{\gamma_i\}_{i=1}^{g} \cup \{\overline{\gamma}_i\}_{i=1}^{g}$ forms a cycle basis for $\mathsf{H}_1(\mathrm{M})$.

First, all elements in $\{[\overline{\gamma}_i]\}_{i=1}^{g}$ are independent and nontrivial because the surface M after cutting along the dual loops $\overline{\gamma}_i$s still remains connected. The Reeb graph for the surface after cutting as it is just the graph after breaking the edges of Rb_{M} at the image points of $\overline{\gamma}_i$s still remains connected. Since the Reeb graph of any function defined on a domain X has the same number of connected components as the domain X, the surface after cutting also remains connected.

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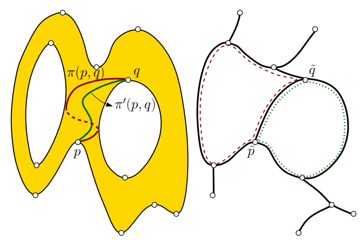
Next, note that the loops $\{\gamma_i\}_{i=1}^g$ are the pre-image loops for $\{c_e\}_{e \in \mathrm{Rb}_\mathrm{M} \setminus T}$ under the surjection $\Phi : \mathrm{M} \to \mathrm{Rb}_\mathrm{M}$, where T is a spanning tree of Rb_M and the latter set $\{c_e\}_{e \in \mathrm{Rb}_\mathrm{M} \setminus T}$ is a basis for cycles in Rb_M. It then follows from Theorem 3.3 from [2] that these loops are nontrivial, independent and generate the so-called *vertical homology group* of $\mathsf{H}_1(\mathrm{M})$. Furthermore, each loop in $\{[\overline{\gamma}_i]\}_{i=1}^g$ is a so-called *horizontal loop* as it lies within a single level set. By the definitions of the vertical and horizontal homology groups (see [1, 2]), the homology classes carried by loops in $\{\gamma_i\}_{i=1}^g$ are independent from those carried by loops in $\{\overline{\gamma}_i\}_{i=1}^g$ in $\mathsf{H}_1(\mathrm{M})$. Hence, the $2g$ independent loops $\{\gamma_i\}_{i=1}^g \cup \{\overline{\gamma}_i\}_{i=1}^g$ form a cycle basis for $\mathsf{H}_1(\mathrm{M})$.

1.3 Implementation details of perturbation of loops (Step 4)

In our implementation, we *do not* physically perturb the entire curve γ_i or $\overline{\gamma}_i$ to get α_i or $\overline{\alpha}_i$. Rather, we perform the following both for simplicity and robustness. We say a loop $\overline{\gamma}_i$ or $\overline{\alpha}_i$ *horizontal* if it is contained within a level set; and a loop γ_i or α_i *vertical* if it cannot be contained within a level set. When computing the linking number between a vertical loop and a horizontal loop, say $\mathrm{Lk}(\gamma_i, \overline{\alpha}_j)$ or $\mathrm{Lk}(\overline{\gamma}_i, \alpha_j)$, we still use γ_j (resp. $\overline{\gamma}_j$) to represent α_j (resp. $\overline{\alpha}_j$), and only physically offset the edges incident to the intersection point of these two curves. By Property (L1) stated in (**Step 3**), γ_i and $\overline{\gamma}_j$ intersects at most once, and the physical perturbation thus only needs to be performed around this intersection point. Next, by Property (L2), two horizontal loops will be un-linked. Hence, the linking number between two horizontal loops is zero, that is, $\mathrm{Lk}(\overline{\gamma}_i, \alpha_j) = 0$ and $\mathrm{Lk}(\gamma_i, \overline{\alpha}_j) = 0$. What remains is to explain how to compute the linking number between two vertical loops, say $\mathrm{Lk}(\gamma_i, \alpha_j)$, and this case requires more care.



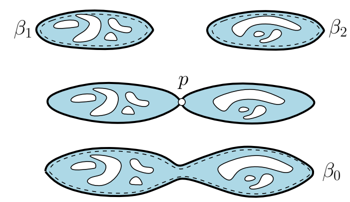
Specifically, recall that γ_i and γ_j come from two loops c_e and $c_{e'}$ in the Reeb graph Rb_M. These two loops could share common arcs in the Reeb graph (such as the red dashed loop and the green dotted loop share $Arc(\tilde{p}, \tilde{q})$ in the right figure). In (**Step 3**), when we map loops c_e and $c_{e'}$ back to M, the common arcs will be mapped to the same sub-curves connecting two critical points, say both are mapped back to $\pi(p, q)$ in the right picture. If we use γ_j to represent α_j, then it is not clear how to resolve such common sub-curves between γ_i and γ_j when computing their linking number. To address this issue, in (**Step 3**), when we map Reeb graph loops back to the surface, we compute two pre-image curves $\pi(p, q)$ and $\pi'(p, q)$ on the surface M for every Reeb graph arc $Arc(\tilde{p}, \tilde{q})$ such that $\pi(p, q)$ and $\pi'(p, q)$ are disjoint except at the end-points p and q. See the figure above for an illustration. For every Reeb graph loop c_e, we now compute and store *two* versions of its pre-image loop on M: γ_i using the pieces $\pi(p, q)$s, and γ'_i using the pieces $\pi'(p, q)$s for each arc $Arc(\tilde{p}, \tilde{q})$ involved. Note that γ_i and γ'_i are disjoint other than at critical points of h, and they are isotopic on M.

Now, instead of computing $\mathrm{Lk}(\gamma_i, \alpha_j)$, we compute $\mathrm{Lk}(\gamma'_i, \alpha_j)$. Since γ_i is isotopic to γ'_i in M and $\alpha_j \cap \mathrm{M} = \emptyset$, it turns out that we can show $\mathrm{Lk}(\gamma_i, \alpha_j) = \mathrm{Lk}(\gamma'_i, \alpha_j)$. Finally, to compute $\mathrm{Lk}(\gamma'_i, \alpha_j)$, we still use γ_j to represent α_j. Now γ_j can only intersect γ'_i potentially at critical points of h on M, and we can resolve such intersections by offsetting the two curves around them locally as before.

2 Proof of Theorem 3.1

Recall that p is a splitting saddle in M. After crossing the critical point p, all other contours remain unchanged except β_0, which splits into β_1 and β_2 in the level set just above $h(p)$. Let P_a denote the plane of the level set at the height value $a \in \mathbb{R}$. By Jordan curve theorem, there are altogether 4 configurations (1.a), (1.b), (2.a) and (2.b) depending on the containment relations of the curves and the inside/outside of M, which is described in the main paper. Furthermore, we note that case (1.a) is equivalent to (2.b) and case (1.b) is equivalent to (2.a) if we compactify $\mathbb{R}^3$ to the 3-sphere $\mathbb{S}^3$ (in which case the level set for the ambient space height function will be the 2-sphere $\mathbb{S}^2$). Hence, we only need to prove the theorem for cases (1.a) and (1.b). Below we consider only the case (1.a): the argument for case (1.b) is symmetric by swapping the role of $\mathbb{I}$ with $\mathbb{O}$.

Proof that γ_i is nontrivial in $\mathsf{H}_1(\mathbb{I})$. Consider $h_I : \mathbb{I} \to \mathbb{R}$ which is the restriction of the ambient space height function to the interior $\mathbb{I}$ of M. Let Rb_I denote the Reeb graph of h_I for $\mathbb{I}$.



Since each β_i' is in $\mathbb{I}$, it means that β_0 is part of the boundary of a region $R_0 \subseteq R[\beta_0]$ and R_0 is a connected component in the level set $h_I^{-1}(a)$ for $a = h(p) - \varepsilon$. Afterwards, this region is split into two regions $R_1 \subseteq R[\beta_1]$ and $R_2 \subseteq R[\beta_2]$ in the level set $h_I^{-1}(b)$ for $b = h(p) + \varepsilon$, where $\beta_1 \subseteq \partial R_1$ and $\beta_2 \subseteq \partial R_2$. See the right figure for an illustration, where the shaded regions in the bottom and top levels are R_0, R_1, and R_2, respectively, and each of these regions is a connected component for some level set of the function h_I.

In other words, p is mapped to an up-fork saddle p' in Rb_I as well. Let p_1' and p_2' be the two regular points in Rb_I that R_1 and R_2 are mapped to. Now consider the map $g : \mathrm{M} \hookrightarrow \mathbb{I} \to \mathrm{Rb}_I$ where the first map $\mathrm{M} \hookrightarrow \mathbb{I}$ is inclusion and the second one is the surjection from the domain $\mathbb{I}$ to its Reeb graph Rb_I. Obviously, $g(\gamma_i)$ is a loop in Rb_I with p' being the lowest point. Since p_1' and p_2' are both in $g(\gamma_i)$, $g(\gamma_i)$ is non-trivial in Rb_I. Hence, it follows from Theorem 3.3 of [2], γ_i is non-trivial in $\mathbb{I}$.

Proof that $\overline{\gamma}_i$ is nontrivial in $\mathsf{H}_1(\mathbb{O})$. We have already shown that γ_i is nontrivial in $\mathsf{H}_1(\mathbb{I})$. We now argue that its dual $\overline{\gamma}_i$ is non-trivial in $\mathsf{H}_1(\mathbb{O})$. Specifically, imagine that we deform $\overline{\gamma}_i$ inside the open exterior $\mathring{\mathbb{O}}$ of M to $\overline{\ell}$ where all points in $\overline{\ell}$ have the same height as points in $\overline{\gamma}_i$ (i.e, we deform $\overline{\gamma}_i$ within the level set of the height function $h_{\mathbb{O}} : \mathbb{O} \to \mathbb{R}$). Similarly, also deform γ_i inside the open interior $\mathring{\mathbb{I}}$ of M to ℓ. Obviously, ℓ and γ_i are homologous in $\mathbb{I}$ while $\overline{\ell}$ and $\overline{\gamma}_i$ are homologous in $\mathbb{O}$. By the construction of $\overline{\gamma}_i$, it intersects γ_i exactly once. Hence, after expanding $\overline{\gamma}$ "outwards" to $\overline{\ell}$ and shrinking γ_i "inwards" to ℓ, we have that the linking number $\mathrm{Lk}(\ell, \overline{\ell}) = 1$. (To see this, note that the loop ℓ now intersects the region bounded by $\overline{\ell}$ within the level set of the ambient height function exactly once.) It then follows from our discussion below Definition 2.2 that the homology class carried by $\overline{\ell}$ is non-trivial in $\mathsf{H}_1(\mathbb{R}^3 - \ell)$. This means that $\overline{\ell}$ is non-trivial in $\mathbb{O} \subset \mathbb{R}^3 - \ell$ as well; otherwise, the image of $[\overline{\ell}]_{\mathbb{O}}$ in $\mathsf{H}_1(\mathbb{R}^3 - \ell)$, under the homomorphism $i_* : \mathsf{H}_1(\mathbb{O}) \to \mathsf{H}_1(\mathbb{R}^3 - \ell)$ induced by inclusion i, will be zero, which contradicts that $\overline{\ell}$ is non-trivial in $\mathsf{H}_1(\mathbb{R}^3 - \ell)$.

3 Proof of Theorem 3.2

Proof of Claim (i). We prove that loops in $A_1 = \{\alpha_{\pi_j}\}_{j=1}^n \cup \{\overline{\alpha}_{\pi_j}\}_{j=n+1}^g$ are all independent in $\mathbb{I}$. The proof for loops in A_2 is symmetric. Since each α_i (resp. $\overline{\alpha}_j$) is homologous to γ_i (resp. $\overline{\gamma}_j$) in $\mathbb{I}$ for $\alpha_i \in A_1$ (resp. $\overline{\alpha}_j \in A_1$), we will show the independence for loops in Γ_1 in $\mathbb{I}$ instead (recall that A_1 contains the perturbation of loops in Γ_1). Note, by Theorem 3.1, we already know that each loop in Γ_1 is non-trivial in $\mathbb{I}$.

Similar to the proof in Section 2, let Rb_I denote the Reeb graph of the height function $h_I : \mathbb{I} \to \mathbb{R}$. Recall that each $\gamma_i \in \Gamma_1$ with the lowest point p is mapped to a loop γ_i' in Rb_I with the same lowest point p' (which is not necessarily true if $\gamma_i \notin \Gamma_1$). Since all γ_is have *distinct* lowest points, the images of $\{\gamma_{\pi_j}\}_{j=1}^n$ are loops with distinct lowest points in Rb_I, and thus they are necessarily independent in Rb_I. By the relationship between the homology group $\mathsf{H}_1(\mathrm{Rb}_\mathsf{I})$ and the so-called vertical homology group $\mathsf{H}_1^{\mathsf{V}}(\mathbb{I})$ for $\mathbb{I}$ (see Theorem 3.3 of [2]), this implies that $\{[\gamma_{\pi_j}]\}_{j=1}^n$ are also independent in $\mathsf{H}_1(\mathbb{I})$.

Next, we observe that each loop in $\{\overline{\gamma}_{\pi_j}\}_{j=n+1}^g$ carries a horizontal homology class in $\mathsf{H}_1(\mathbb{I})$, as it is contained within a level set for h_I. Each loop in $\{\gamma_{\pi_j}\}_{j=1}^n$ on the other hand, carries a vertical homology class, which means that no loop homologous to them can be horizontal. Thus loops in $\{\gamma_{\pi_j}\}_{j=1}^n$ are independent from loops in $\{\overline{\gamma}_{\pi_j}\}_{j=n+1}^g$.

What remains is to show that loops in $\{\overline{\gamma}_{\pi_j}\}_{j=n+1}^g$ are also independent in $\mathbb{I}$. To this end, we will change back to argue it for loops $\{\overline{\alpha}_{\pi_j}\}_{j=n+1}^g$. Assume that a subset of these loops generates a trivial homology class in $\mathsf{H}_1(\mathbb{I})$, that is, suppose that $[\overline{\alpha}_{s_1}]+\ldots+[\overline{\alpha}_{s_r}] = 0$ in $\mathsf{H}_1(\mathbb{I})$ for some indices $s_1,\ldots,s_r \in \{\pi_j\}_{j=n+1}^g$. Each of these loops is contained within a level set, and assume w.o.l.g. that $\overline{\alpha}_{s_r}$ has the highest height among them. Now consider the dual loop α_{s_r} of $\overline{\alpha}_{s_r}$: recall by construction, $\alpha_{s_r} \in \mathring{\mathbb{O}}$. Since each $\overline{\alpha}_{s_i}$ is lower than the lowest point of α_{s_r} (which has the same height as $\overline{\alpha}_{s_r}$), $\mathrm{Lk}(\alpha_{s_r}, \overline{\alpha}_{s_i}) = 0$ for $i \in [1, r-1]$ while $\mathrm{Lk}(\alpha_{s_r}, \overline{\alpha}_{s_r}) = 1$. This means that $[\overline{\alpha}_{s_1}]+\ldots+[\overline{\alpha}_{s_r}]$ is non-trivial in $\mathsf{H}_1(\mathbb{R}^3 - \alpha_{\mathsf{s_r}})$. Similar to the argument used in the second part of the proof in Section 2, this then implies that $[\overline{\alpha}_{s_1}]+\ldots+[\overline{\alpha}_{s_r}]$ cannot be trivial in $\mathbb{I} \subset \mathbb{R}^3 - \alpha_{s_r}$, which contradicts our assumption. Hence, our assumption is wrong and loops in $\{\overline{\alpha}_{\pi_j}\}_{j=n+1}^g$ are independent. Putting everything together, we have that loops in $A_1 = \{\alpha_{\pi_j}\}_{j=1}^n \cup \{\overline{\alpha}_{\pi_j}\}_{j=n+1}^g$ are all independent in $\mathbb{I}$. Finally, since the rank of $\mathsf{H}_1(\mathbb{I})$ is g, we have that A_1 forms a cycle basis for $\mathsf{H}_1(\mathbb{I})$.

Proof of Claim (ii). Note that each index i, $i \in [1, n]$, appears exactly once in A_1 (and exactly once in A_2) either as α_i or as $\overline{\alpha}_i$. So we can permute the matrix D so that the i-th row contains the linking number between either the α_i or $\overline{\alpha}_i$ (depending on which one is in A_1) and all elements in A_2; while the i-th column contains the linking number between either $\overline{\alpha}_i$ or α_i and all elements in A_1. It is clear that this new matrix is non-singular if and only if D is non-singular. For simplicity, we call the new matrix also D.

Now consider $i = 1$. Suppose that $\overline{\alpha}_1 \in A_1$. Then the first row contains the linking number between $\overline{\alpha}_1$ and all loops in A_2. Recall that loops $\gamma_1, \ldots, \gamma_g$ (and thus $\alpha_1, \ldots, \alpha_g$) are sorted by the height of their lowest point. Hence, $\overline{\alpha}_1$ has the lowest height. It is thus unlinked with α_j and $\overline{\alpha}_j$ for $j \geq 2$, and by construction, $\mathrm{Lk}(\overline{\alpha}_1, \alpha_1) = 1$. This means that the first row of D is $[1, 0, \ldots, 0]_g$. If $\alpha_1 \in A_1$, then $\overline{\alpha}_1 \in A_2$, and by the same argument, the first column would be $[1, 0, \ldots, 0]_g^T$. In either case, D is non-singular if and only if the submatrix D_1 generated by removing the first row and first column in D is nonsingular. Note that D_1 now contains only (a subset of) linking numbers between cycles from $\{\alpha_j\}_{j=2}^g \cup \{\overline{\alpha}_j\}_{j=2}^g$.

We now repeat this same procedure. In the ith round, we have the matrix D_i which is obtained by removing the first i number of columns and rows from D. The matrix D is non-singular if and only if D_i is non-singular. We now consider the $i+1$-th row. By Property (L1), (L2) and (L3) introduced in **Step 3**, $\overline{\alpha}_{i+1}$ is unlinked with all α_js and $\overline{\alpha}_j$s for $j > i+1$, and only links with α_{i+1}. Hence, by the same argument as the previous paragraph, we have that the either the first row of D_i is $[1, 0, \ldots, 0]_{g-i}$ or the first column is $[1, 0, \ldots, 0]_{g-i}^T$. In either case, D_i is non-singular if and only if the matrix D_{i+1}, obtained by removing the first column and row from D_i, is non-singular. By repeating this procedure $g-1$ times, we reach D_{g-1} which is of size 1 and has value 1 (since $\mathrm{Lk}(\alpha_g, \overline{\alpha}_g) = 1$). Hence, D_{g-1} is non-singular, implying that any D_i for $i < g-1$, including $D_0 = D$, is non-singular.

4 Proof of Lemma 3.3

We will prove the first half of the lemma, and the proof for the second half is symmetric. First, note that if a loop ℓ forms a non-zero linking number with any loop $\alpha \in A_2$, then ℓ carries a non-trivial homology class in $\mathsf{H}_1(\mathbb{I})$. This is because that for a loop $\alpha \in A_2$, we have that $\alpha \subset \mathring{\mathbb{O}}$. Hence, if $[\ell]$ is trivial in $\mathsf{H}_1(\mathbb{I})$, then under the homomorphism induced by inclusion $\mathbb{I} \hookrightarrow \mathbb{R}^3 - \alpha$, $[\ell]$ will be trivial in $\mathsf{H}_1(\mathbb{R}^3 - \alpha)$ as well. In other words, $\mathrm{Lk}(\alpha, \ell) = 0$ which is a contradiction. Hence, ℓ is nontrivial in $\mathsf{H}_1(\mathbb{I})$.

Next, we show that if ℓ forms zero linking number with every loop of A_1, then ℓ is necessarily trivial in $\mathsf{H}_1(\mathbb{O})$. We prove this by contradiction. Assume that $[\ell]$ is non-trivial in $\mathsf{H}_1(\mathbb{O})$. Rewrite $A_2 = \{\beta_1, \ldots, \beta_g\}$ for simplicity. Since A_2 forms a cycle basis for $\mathsf{H}_1(\mathbb{O})$, we have that $[\ell] = [\beta_{s_1}] + \cdots + [\beta_{s_r}]$ for some $s_i \in [1, g]$ and $r \leq g$. On the other hand, since $\mathrm{Lk}(\ell, \alpha) = 0$ for each $\alpha \in A_1$, this means that $\mathrm{Lk}(\ell, \alpha) =$

$\sum_{i=1}^{r} \mathrm{Lk}(\beta_{s_i}, \alpha) = 0$ for every $\alpha \in A_1$. In other words, consider the matrix D of linking numbers as introduced in Theorem 3.2, the columns corresponding to β_{s_i}, $i \in [1, r]$ are linearly dependent. This is a contradiction, as the matrix D is non-singular. We thus have that ℓ must carry a trivial homology class in $\mathsf{H}_1(\mathbb{O})$.

References

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