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Triangulations

Organised by
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April 29th – May 5th, 2012

Abstract. The earliest work in topology was often based on explicit combinatorial models – usually triangulations – for the spaces being studied. Although algebraic methods in topology gradually replaced combinatorial ones in the mid-1900s, the emergence of computers later revitalized the study of triangulations. By now there are several distinct mathematical communities actively doing work on different aspects of triangulations. The goal of this workshop was to bring the researchers from these various communities together to stimulate interaction and to benefit from the exchange of ideas and methods.

Mathematics Subject Classification (2000): 52B, 57M (Primary); 05A, 05E, 11H, 14M, 14T, 52C, 53A, 57N, 57Q, 68U (Secondary).

Introduction by the Organisers

The workshop *Triangulations*, organised by William H. Jaco (Stillwater), Frank H. Lutz (Berlin), Francisco Santos (Santander) and John M. Sullivan (Berlin) was held April 29th – May 5th, 2012. The meeting was well attended, with 53 participants from 14 countries (including Argentina, Australia and Israel). Besides the 27 lectures, the program included evening sessions on mathematical software and on open problems. The workshop successfully brought the different communities of mathematicians interested in triangulations together, resulting in several new collabortive projects between mathematicians who had never met before.

Triangulations have become increasingly important in both discrete geometry and manifold topology, but this work has proceeded independently without much interaction between the communities. Even the word "triangulation" can be a source of confusion: for 3-manifolds the most general pseudo-simplicial triangulations (typically with a single vertex) are preferred, but discrete geometers mostly restrict to simplicial complexes, while in polytope theory and computational geometry these must be linearly embedded. Thus for instance while Pachner moves (bistellar flips) are useful to pass from one triangulation to another (as highlighted in the talks by de Loera and Burton), the moves available and their exact properties depend on the class of triangulations considered.

The combinatorial approach through 0-efficient triangulations and normal surfaces has introduced methods from geometric analysis into the combinatorial study of 3-manifolds. The resulting enumeration and decision problems are important and challenging examples in the study of computational complexity (as shown in the talks by Schleimer and Hass).

Complexity measures for 3-manifolds are well established in the pseudo-simplicial world (see the talks by Matveev, Martelli, Casali and Tillmann), but there is also a possible analogue in the combinatorial triangulation world (as proposed in the talk by Swartz).

Within discrete geometry, tools of algebraic topology and even algebraic geometry are often essential for answering fundamental questions. For instance, Stanley's proof of the g-theorem (characterizing the f-vectors of polytopes) is based on the cohomology of toric varieties. Recent progress along these lines was reported by Joswig, Nevo and Swartz.

One example of a technique used in both communities to relate geometry and combinatorics is putting CAT(0) metrics on triangulations, as mentioned in the talks by Rubinstein, Benedetti and Adiprasito.

The workshop schedule left plenty of free time for informal interactions, and many fruitful discussions developed between mathematicians who had just met for the first time. For instance, knowledge about simplicial decompositions of the dodecahedron led to new insights on minimal triangulations of the Seifert–Weber dodecahedral space. As another example, less than two months after the workshop, Hähnle, Klee and Pilaud posted a preprint (arXiv:1206.6143) on weak decomposability based on work started at the Oberwolfach workshop.

Workshop: Triangulations

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3-Manifold Recognizer and 3-Manifold Atlas

Sergei Matveev
(joint work with V. Tarkaev, E. Fominykh, Ph. Korablev, V. Potapov, E. Sbrodova, A. Kazakov, D. Gorkovez, and other members of the topology group of Chelyabinsk State University)

3-Manifold Recognizer is a huge computer program for recognizing closed orientable 3-manifolds. It accepts almost all known representations of 3-manifolds: genuine triangulations, one-vertex triangulations, special spines, representations by surgeries along framed links, crystallizations, genus two Heegaard diagrams, and some other representations.

The output of the Recognizer is the *name* of the given manifold M. Possible names are: complete information on the Seifert or graph-manifold structure (if M is Seifert or graph-manifold of Waldhausen [1]), a surgery description of M as well as a representation of M in the form of a Dehn filling of an elementary brick from [2]. If M is geometric, we get the type of the geometry, including the volume (if M is hyperbolic) and the monodromy matrix (if M possesses Sol geometry). If M is composite, the name is a description of the JSJ-decomposition of M (including information about the types of JSJ-chambers and gluing homeomorphisms). The name *Unrecognized* is also possible. In all cases we get values of different invariants of the given manifold: the type of geometry, values of Turaev-Viro invariants (up to order 16, if you are patient enough), of course, first homology group, and some additional information.

The Recognizer works as follows: given M, the computer creates a special spine P of M. Then it tries to simplify P by different moves, which may change M and P, but only in controlled manner. For example, the first move may consist of removing a 2-cell C from P. As the result of this move, we get a simpler manifold M_0 (whose boundary is a torus), and its spine $P_0 = P \setminus C$. Of course, P_0 is not special anymore, thus the computer have to keep in memory information how P_0 can be thickened to M_0 and how M can be obtained from M_0 by attaching a solid torus. The crucial advantage of this approach is that after admitting general (not necessary special) spines we get a big freedom for working with them. Other moves include cutting M along proper discs and annuli, and corresponding moves on P. As the result of those moves we get a decomposition of M into a union of several pieces with boundaries consisting of tori, and information how these pieces can be assembled into the original manifold M. The pieces are called *atoms*, and the information on the assembling is called a *molecule*. During the assembling process the program keeps track of the topology of 3-manifolds obtained at each step until getting the final result. See [3], Chapter 7.

The Recognizer turned out to be a very powerful tool for recognition and tabulation of 3-manifolds. Using it, we composed the following table.

c	≤ 5	6	7	8	9	10	11	12	13	Total
N	61	74	175	436	1154	3078	8421	23448	66198	103042

Here c is the complexity and N is the number of closed orientable irreducible 3-manifolds of complexity c.

Let us give a non-formal description of the computer program that was used for creating the table. The computer enumerates all regular graphs of degree 4 with a given number of vertices. The graphs may be considered as work-pieces for singular graphs of special spines. For each graph, the computer replaces each vertex by a butterfly (typical neighborhood of a true vertex of a special polyhedron) and enumerates all possible gluings of those butterflies along the edges of the graph. Then 2-cells are attached. If the special polyhedron thus obtained is a spine of a closed 3-manifold, we send it to the Recognizer and include it into the table, if the manifold turns out to be a new one. An independent check of the table by computing Turaev-Viro invariants up to order 16 was very useful for removing duplicates. Of course, all reasonable tricks for accelerating the process had been implemented. We used the supercomputer of South-Ural State University (that time it was the third in Russia and among first 50 in the world). However, the running time of the program was about a year (with a few interruptions).

3-Manifold Atlas is the interactive site **www.matlas.math.csu.ru** based on the above table and equipped with a search function. If you specify information on a manifold your are looking for, you get the list of all manifolds from the table which satisfy your criteria. Clicking at any of those manifolds, you get additional information including all above-mentioned invariants and the standard code of one of its minimal special spines.

Acknowledgments The topology group of Chelyabinsk State University was partially supported by Russian Fund of Basic Research, Mathematical and Ural branches of RAS, and by Institute of Mathematics and Mechanics of Ural branch of RAS.

We thank Ben Burton for pointing out two errors in the above table.

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On Spaces of Triangulations of Convex Polytopes and Point Configurations

Jesús A. De Loera

In the late 1980's and early 1990's there was a lot of work trying to understand the structure of the set of all triangulations of a convex polytope (e.g., the unit cube) or a configuration of points in R^d. The prime example is the set of triangulations of an n-gon, which gives rise to the well-known *associahedron*. Using a clever construction Gelfand, Kapranov, and Zelevinsky [1] showed that this example generalizes. To all *regular* subdivisions of a set of n points $a_1, \ldots, a_n$ in R^d one can associate an $(n-d-1)$-dimensional convex polytope whose faces are in bijection with the regular subdivisions of the point set. Its vertices correspond to the so-called regular triangulations of the point set. The flip graph of a point configuration is the graph whose vertices are the triangulations of the configuration, and where two vertices are connected by an edge if there exists a bistellar operation transforming one triangulation into the other. The 1-skeleton of the secondary polytope is a subgraph of the flip graph induced by the regular triangulations of the point configuration (but recently it has been shown that not all flips appear as edges [2]). My presentation aimed to introduced the audience to these ideas (which contrast with the use of triangulations in 3-dimensional topology). I concluded with several open questions that still motivate active research. We present this collection of questions here, which is a subset of those presented in the book [2].

(1) Can we find all the facets of the secondary polytope of the d-cube?
(2) It is well known that in dimension 2 the graph of *all* triangulations is connected. But Santos proved [3] that in dimension 5 or higher, there are point sets with a disconnected graph of triangulations [Santos, 2004]. Are there disconnected 3-dimensional examples? How about dimension 4?
(3) What is the computational complexity of counting **all** triangulations? How to generate a triangulation uniformly at random?
(4) Prove/Disprove that all *smooth polytopes* have a unimodular regular triangulation.
(5) Prove/Disprove that all matroid polytopes have a unimodular regular triangulation.
(6) Is it true that every 3-dimensional polyhedron has a triangulation whose dual graph is Hamiltonian?
(7) What are the best bounds for the diameter of a triangulation?

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Pachner Moves, Generic Complexity, and Randomising 3-Manifold Triangulations

Benjamin A. Burton
(joint work with Murray Elder, Jonathan Spreer and Stephan Tillmann)

1. Introduction

We study the computational complexity of decision problems on triangulated 3-manifolds. In this setting there has been encouraging initial progress in recent years, but many important questions remain wide open.

The "simple" problem of 3-sphere recognition and the related problem of unknot recognition are both known to be in NP, by work of Schleimer [18] and earlier work of Hass, Lagarias and Pippenger [9] respectively. In recent announcements by Kuperberg [13] and Hass and Kuperberg [8], these problems are also in co-NP if the generalised Riemann hypothesis holds. It remains a major open question as to whether either problem can be solved in polynomial time.

There are very few hardness results for such problems. A notable example due to Agol, Hass and Thurston is knot genus: if we generalise unknot recognition to computing knot genus, *and* we generalise the ambient space from S^3 to an arbitrary 3-manifold, then the problem becomes NP-complete [1]. The key construction in their result can also be adapted for problems relating to least-area surfaces [1, 7].

2. A hardness result: Taut angle structures

Our first result (in joint work with Spreer) is a hardness result. It relates to taut angle structures on triangulations, as introduced by Lackenby [14]. Taut angle structures are simple and common combinatorial objects; in the right settings they can lead to strict angle structures [11], which are richer objects that in turn can point the way towards building complete hyperbolic structures.

We use the nomenclature of Hodgson et al. [10]: a *taut angle structure* assigns interior angles $\{0,0,0,0,\pi,\pi\}$ to the six edges of each tetrahedron of a triangulation, so that the two π angles are opposite in each tetrahedron, and so that around each edge of the overall triangulation the sum of angles is 2π. Note that this requires the triangulation to be ideal, with torus or Klein bottle vertex links. These structures are slightly more general than the taut structures of Lackenby [14], who also requires consistent coorientations on the 2-faces of the triangulation.

The decision problem that we study is a simple one:

Problem 1 (TAUT ANGLE STRUCTURE)**.** *Given an orientable 3-manifold triangulation $\mathcal{T}$ as input, determine whether there exists a taut angle structure on $\mathcal{T}$.*

This decision problem explicitly asks about the geometry of the input triangulation, not the underlying manifold. Our main result is the following:

Theorem 1. TAUT ANGLE STRUCTURE *is NP-complete.*

The proof uses a reduction from MONOTONE 1-IN-3 SAT, which was shown by Schaefer to be NP-complete in the 1970s [17]. In MONOTONE 1-IN-3 SAT we have boolean variables $x_1, \ldots, x_t$ and clauses of the form $x_i \vee x_j \vee x_k$, and we must determine whether the t variables can be assigned true/false values so that *precisely* one of the three variables in each clause is true.

For any instance $\mathcal{M}$ of MONOTONE 1-IN-3 SAT, we build a corresponding triangulation that has a taut angle structure if and only if $\mathcal{M}$ is solvable. The triangulation is built by hooking together three types of gadgets: (i) *variable gadgets*, each with two choices of taut angle structure that represent true or false respectively for a single variable x_i of $\mathcal{M}$; (ii) *fork gadgets* that allow us to propagate this choice for x_i to several clauses simultaneously; and (iii) *clause gadgets* that connect three variable gadgets and support an overall taut angle structure if and only if precisely one of the three corresponding variable choices is true. These gadgets have 2, 21 and 4 tetrahedra respectively, and were constructed with significant assistance from the software package *Regina* [2, 5].

This result offers a new framework for proving NP-completeness by building up concrete 3-manifold triangulations, and it is an ongoing project to see how far we can push this framework towards key decision problems such as 0-efficiency testing, 3-sphere recognition, and unknot recognition.

3. Towards an easiness result: 3-sphere recognition

We turn our attention now to 3-sphere recognition. Here we offer a framework (in joint work with Elder and Tillmann), supported by empirical evidence, for solving 3-sphere recognition in polynomial time for *generic* 3-sphere triangulations.

The current state of the art for 3-sphere recognition is outlined in [3] (a cumulation of many results by many authors), and has a running time of $O(7^n \cdot \mathrm{poly}(n))$ in the worst case [6] (Casson describes an $O(3^n \cdot \mathrm{poly}(n))$ solution, but his method is not practical because the 3^n factor is largely unavoidable). Nevertheless, practical implementations are extremely fast in practice: Regina takes just 0.25 milliseconds on average to recognise a 3-sphere triangulation with $n = 10$ tetrahedra.

The key to this speed is *simplification*: Regina will first try to greedily reduce the input triangulation to a small number of tetrahedra using Pachner moves (bistellar flips) [15, 16]; for most 3-sphere triangulations this yields a one-tetrahedron triangulation that can be recognised immediately without running the expensive $O(7^n \cdot \mathrm{poly}(n))$ algorithm at all.

We seek to capture this behaviour using *generic complexity* [12], which allows us to exclude rare pathological inputs from consideration. Let I_n denote all possible inputs of size n. A set of inputs S is called *generic* if $|S \cap I_n|/|I_n| \to 1$ as $n \to \infty$; in other words, the inputs excluded from S become "infinitesimally rare".

We analyse the census of all $31,017,533$ one-vertex 3-sphere triangulations with $n \leq 9$ tetrahedra, and measure paths of Pachner moves between them [4]. Let $p_{n,k}$ denote the probability that a random n-tetrahedron one-vertex 3-sphere triangulation *cannot* be simplified in $\leq k$ such moves. Empirically $p_{n,k}$ falls very fast, at a rate that appears comfortably $O(1/\alpha^{nk})$ for fixed $\alpha > 1$.

Under the right "approximate independence" assumptions, this would allow us to simplify *generic* one-vertex 3-sphere triangulations to a known two-tetrahedron 3-sphere triangulation in polynomial time. The key idea is "aggressive simplification": as our triangulation shrinks we gradually increase the number of allowed moves, allowing us to "jump past" smaller difficult cases as $n \to \infty$, but maintaining an overall polynomial running time of bounded degree in n.

This work ties into the study of random 3-manifold triangulations, where very little is known. Understanding random triangulations—and how to effectively randomise a triangulation of an arbitrary 3-manifold—is an ongoing challenge with significant implications for topological algorithms and complexity.

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Triangulations of Products of Simplices with a View Towards Tropical Geometry

MICHAEL JOSWIG
(joint work with Sven Herrmann, David Speyer)

One goal of tropical geometry is to address certain problems in algebraic geometry by means of geometric combinatorics. Conversely, classical problems in combinatorial optimization receive a natural geometric interpretation. Here we give an example of this relationship related to the structure of the tropical Grassmannians introduced by Speyer and Sturmfels [6]. On the combinatorial level this approach works via lifting subdivisions of products of simplices to matroid decompositions of hypersimplices.

Let Δ_{d-1} be a $(d-1)$-dimensional simplex. The actual shape does not matter. The triangulations of the *prism* $\Delta_{d-1} \times [0,1]$ have a particularly simple structure. If we label the vertices of Δ_{d-1} by the elements in the set $[d] := \{1, 2, \ldots, d\}$ then any triangulation Γ of $\Delta_{d-1} \times [0,1]$ correspond to the permutations of the set $[d]$. Moreover, Γ is always regular, and the dual graph is a path on d nodes; see [1, §6.2]. In fact, the normal fan of the *permutahedron*

$$\operatorname{conv}\{(\omega(1), \omega(2), \ldots, \omega(d)) \mid \omega \in \operatorname{Sym}(d)\} \subset \mathbb{R}^d$$

is the secondary fan of the prism $\Delta_{d-1} \times [0,1]$; that is, this fan stratifies the lifting functions by combinatorial type of the induced subdivision. Here $\operatorname{Sym}(d)$ denotes the set of permutations of the set $[d]$.

Products of simplices occur in tropical geometry via a result of Develin and Sturmfels [2]: The dual polytopal complex (also called *tight span*) of the regular subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ induced by the lifting function $V \in \mathbb{R}^{d\times(n-d)}$ is isomorphic to the natural polytopal subdivision of the *tropical convex hull* of the columns (or rows) of the matrix V. Triangulations correspond to matrices which are sufficiently generic. We want to relate the secondary fan of $\Delta_{d-1} \times \Delta_{n-d-1}$ to the *tropical Grassmannian* $\operatorname{Gr}_K(d,n)$. The latter is defined as the tropical variety of the Plücker ideal in the polynomial ring $K[x_\sigma | \sigma \in \binom{n}{d}]$ over some field K. Notice, that for products of simplices other than prisms the precise structure of the secondary fans is rather involved and unknown, except for a few special cases.

By E_d we denote the *tropical identity matrix* of rank d (the $d\times d$-matrix with 0 on the diagonal and coefficients equal to ∞ otherwise). For a matrix $V \in \mathbb{R}^{d\times(n-d)}$ we let $\bar{V} = (E_d|V)$ be the $d\times n$-matrix arising by block column concatenation of E_d and V. Each d-element subset $\sigma \subseteq [n]$ specifies a $d\times d$-submatrix $\bar{V}_\sigma$ by selecting the columns of $\bar{V}$ whose indices are in σ. We are interested in the map

$$\tau_V : \binom{[n]}{d} \to \mathbb{R}, \quad \sigma \mapsto \operatorname{tdet}(\bar{V}_\sigma). \tag{1}$$

Here

$$\operatorname{tdet}(A) = \min_{\omega\in\operatorname{Sym}(d)} a_{1,\omega(1)} + a_{2,\omega(2)} + \cdots + a_{k,\omega(k)} \tag{2}$$

denotes the *tropical determinant* of the matrix $A = (a_{ij})_{i,j} \in \mathbb{R}^{d\times d}$.

It is readily verified that the map τ_V is a *tropical Plücker vector* or, equivalently, a lifting function of the hypersimplex

$$\Delta(d,n) \ = \ \operatorname{conv}\left\{ e_\sigma \ \middle| \ \sigma \in \binom{[n]}{d} \right\} \ \subset \ \mathbb{R}^{\binom{n}{d}}$$

which induces a matroid decomposition [3]; here e_σ is the 0/1-vector of length n whose d ones correspond to the elements of σ. A *matroid decomposition* is a polytopal subdivision of $\Delta(d,n)$ such that the vertices of each cell correspond to the bases of a matroid. Equivalently, it is a subdivision of $\Delta(d,n)$ without new edges. Moreover, τ_V is the tropicalization of a classical Plücker vector, whence it is contained in $\mathrm{Gr}_K(d,n)$. Notice that this is independent of the field K.

Theorem 1 ([4])**.** *The map $\tau : V \mapsto \tau_V$ is a piecewise-linear embedding of the secondary fan of $\Delta_{d-1} \times \Delta_{n-d-1}$ into $\mathrm{Gr}_K(d,n)$ which preserves the tight spans.*

A few remarks are in order. First, this is a strengthening of a result of Kapranov [5, 4.1.4]. Second, the embedding is into $\mathrm{Gr}_K(d,n)$ *as a set*; that is, we do not fully control how τ behaves with respect to any fan structure on the tropical Grassmannian. Third, a similar result has been obtained independently by Rincón[1]. Fourth, our result generalizes to non-regular subdivisions of products of simplices: They, too, lift to (non-regular) matroid subdivisions of the suitable hypersimplex.

We conclude by returning to the initial discussion of the triangulations of prisms. In this case (where $n = d + 2$) our theorem above says the following. Suppose $V \in \mathbb{R}^{d\times 2}$ is a generic lifting function on the prism $\Delta_{d-1} \times [0,1]$. Then τ_V is a lifting function of $\Delta(d, d+2)$ which induces a matroid subdivision whose tight span is a path of length d. This tight span coincides with the tropical convex hull of the two columns of the matrix V, a tropical line segment. The tropical Grassmannian $\mathrm{Gr}_K(2,n)$ is the space of all trivalent trees with n labeled nodes. The image of the map τ in $\mathrm{Gr}_K(d, d+2) \cong \mathrm{Gr}_K(2, d+2)$ is the set of *caterpillar trees*.

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Triangulations and Subdivisions in the Geometry of Numbers

Frank Vallentin

1. Classical theory

The classical theory of triangulations and subdivisions in the geometry of numbers goes back to G.F. Voronoi (1868–1908) and B.N. Delaunay = Б.Н. Делоне = B.N. Delone (1890–1980). It can be seen as a predecessor of Gel'fand, Kapranov, Zelevinsky's theory of regular triangulations.

In the classical theory one considers periodic regular subdivisions whose vertex set is the lattice $\mathbb{Z}^n$ and where the lifting function which defines the regular subdivision is coming from a positive semidefinite matrix $Q \in \mathcal{S}^n_{\succeq 0}$: The lifting function

$$l_Q : \begin{cases} \mathbb{Z}^n & \to & \mathbb{R}^n \times \mathbb{R}_{\geq 0} \\ v & \mapsto & (v, v^\mathsf{T} Q v) \end{cases}$$

defines (by taking the convex hull of $l_Q(\mathbb{Z}^n)$ and projecting its lower part onto $\mathbb{R}^n$) the *Delone subdivision* $\mathrm{Del}(Q)$.

The *secondary cone* of a Delone subdivision $\mathrm{Del}(Q)$ (traditionally called *L-type domain*) is the set of all Q' for which the subdivision stays constant to $\mathrm{Del}(Q)$,

$$\mathbf{C}^\circ(\mathrm{Del}(Q)) = \{Q' \in \mathcal{S}^n_{\succeq 0} : \mathrm{Del}(Q') = \mathrm{Del}(Q)\}.$$

It is a relatively open polyhedral cone. Cones of full dimension correspond to triangulations. The *secondary fan* is the infinite face-to-face tiling of $\mathcal{S}^n_{\succeq 0}$ by secondary cones. Two Delone triangulations whose secondary cones share a facet differ by flip.

The group $\mathsf{GL}_n(\mathbb{Z})$ acts on $\mathcal{S}^n_{\succeq 0}$. The following finiteness result is the basis of Voronoi's reduction theory.

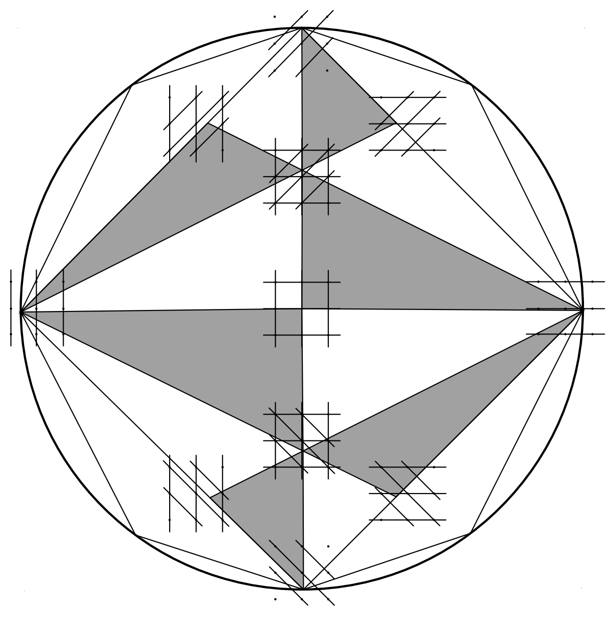

Figure 1. $\mathcal{S}^2_{\succeq 0}/\mathsf{GL}_2(\mathbb{Z})$.

Theorem 1. *There are only finitely many Delone triangulations under the $\mathsf{GL}_n(\mathbb{Z})$-action, so secondary cones give a fundamental domain of $\mathcal{S}^n_{\succeq 0}/\mathsf{GL}_n(\mathbb{Z})$.*

2. Applications and extensions

In a series of papers [1–5] we applied and extended Voronoi's reduction theory to make progress on the *lattice sphere covering problem.* This problem asks for finding a lattice $L = A\mathbb{Z}^n$, with $A \in \mathsf{GL}_n(\mathbb{R})$, so that the density of the sphere covering given by L (spheres of equal size are centered at the lattice points with a radius chosen so they just cover every point in space) is minimized. Going from A to $Q = AA^{\mathsf{T}} \in \mathcal{S}^n_{\succ 0}$ one can formulate the lattice sphere covering problem as the following optimization problem (see [2]):

$$\min_{\mathcal{T}} \max_{\mathcal{T} \leq \mathrm{Del}(Q)} \log\det Q$$
$$\begin{pmatrix} 4 & v_1^{\mathsf{T}}Qv_1 & v_2^{\mathsf{T}}Qv_2 & \dots & v_n^{\mathsf{T}}Qv_n \\ v_1^{\mathsf{T}}Qv_1 & v_1^{\mathsf{T}}Qv_1 & v_1^{\mathsf{T}}Qv_2 & \dots & v_1^{\mathsf{T}}Qv_n \\ \vdots & \vdots & \vdots & & \vdots \\ v_n^{\mathsf{T}}Qv_n & v_n^{\mathsf{T}}Qv_1 & v_n^{\mathsf{T}}Qv_2 & \dots & v_n^{\mathsf{T}}Qv_n \end{pmatrix} \succeq 0, \quad T = \mathrm{conv}\{0, v_1, \dots, v_n\} \subseteq \mathcal{T},$$

where we minimize over all Delone triangulations $\mathcal{T}$. We have one LMI (linear matrix inequality) for every simplex T which describes the condition that the circumradius of the simplex is at most 1. The inner maximization optimization problem is equivalent to a semidefinite optimization problem and thus one can solve it efficiently.

Some words about the complexity of the lattice sphere covering problem: In every Delone triangulation there are up to $n!$ simplices in $\mathrm{Del}(Q)/\mathbb{Z}^n$. Furthermore computing $|\mathrm{Del}(Q)/\mathbb{Z}^n|$ for given Q is #P-hard (see [4]). However, as usual, *symmetry helps*: In [3] we developed and used an equivariant theory for Q's with prescribed automorphism group and we found new record breaking lattice coverings.

Another way of using symmetries is based on spherical t-designs. These are finite point sets on the unit sphere S^{n-1} so that

$$\int_{S^{n-1}} f(x)d\omega(x) = \frac{1}{|X|}\sum_{x\in X} f(x)s$$

holds for all polynomials of degree $\leq t$.

Using this we proved in [1] that the Leech lattice Λ_{24} provides a locally optimal sphere covering. To prove this we remember that Conway, Parker, Sloane (1982) classified Delone polytopes of Λ_{24} with maximum circumradius $\sqrt{2}$ and among them there is the non-regular simplex of type A_{25} where circumcenter and barycenter coincide. Now by dualizing the convex optimization problem and relaxing (only simplices T of type A_{25} are taken into account) and using the spherical 2-design property of the vectors of Λ_{24} of given length one can solve this relaxation by hand, proving local optimality of Λ_{24}.

On the other hand, somewhat unexpected, many highly symmetric lattices do **not** give a good sphere covering as we showed in [5]. In fact, if the covering radius is attained only at non-simplices and if the vertices (after a suitable congruence transformation) of such a Delone polytope form a spherical 2-design, then *almost all* perturbations of the lattice improves the covering density. If they form a spherical 4-design, then *all* perturbations improve the covering density. In Table1 we list the covering property of the most prominent lattices.

lattice	**covering density**
$\mathbb{Z}$	global minimum
A_2	global minimum
D_4	almost local maximum
E_6	local maximum
E_7	local maximum
E_8	almost local maximum
K_{12}	almost local maximum
BW_{16}	local maximum
Λ_{24}	local minimum

Table 1. Covering property of prominent lattices.

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Veering Triangulations and the Cannon–Thurston Map

François Guéritaud

Hyperbolic mapping tori. Let S be an oriented surface with at least one puncture, and $\varphi : S \to S$ an orientation-preserving homeomorphism. Define the *mapping torus* $M := S \times [0,1]/\sim_\varphi$, where $\sim_\varphi$ identifies $(x,0)$ with $(\varphi(x),1)$. The topological type of the 3-manifold M depends only on the isotopy type of φ.

In what follows, we shall assume that φ is *pseudo-Anosov*, a technical condition meaning that the isotopy class $[\varphi]$ preserves no finite system of curves on S. A landmark result of Thurston's [6] is that M then admits a (unique) complete *hyperbolic metric*: $M \simeq \Gamma\backslash\mathbb{H}^3$ for some discrete group of isometries Γ. An important step towards this is the existence of a transverse pair of $[\varphi]$-invariant *singular foliations* λ^+, λ^- of S into lines (called *leaves*).

In [1], Agol described a canonical way of triangulating a mapping torus M, provided all singularities of the foliations λ^+, λ^- occur at punctures of the fiber S.

These (ideal) triangulations enjoy a combinatorial property called *veeringness.* In [5] and [4], veering triangulations are shown to admit *positive angle structures*: this is a linearized version of the problem of finding the complete hyperbolic metric on M (endowed with a geodesic triangulation).

Combinatorics of the veering triangulation. I first presented an alternative construction of Agol's triangulation, which can be summarized as follows. Endow S with a flat (incomplete) metric for which the lines of the measured foliations λ^+ and λ^- are vertical and horizontal, respectively. Look for all possible maximal rectangles $R \subset S$ with edges along leaf segments. By maximality, such a rectangle R contains one singularity in each of its four sides. Connecting these four points and thickening, we get a tetrahedron $\Delta_R \subset S \times [0, 1]$. It only remains to check that the tetrahedra Δ_R glue up to yield a triangulation of $S \times [0, 1]$ (naturally compatible with the equivalence relation $\sim_\varphi$ since the foliations λ^+, λ^- are $[\varphi]$-invariant).

Unlike Agol's original definition, this does not rely on any auxiliary choices (*e.g.* of train tracks). One upshot is that it allows a detailed description of the induced 2-dimensional triangulations $\mathcal{T}$ of the vertex links (which are tori). The details do not matter, but each torus turns out to be decomposed into an even number of parallel annuli, with each triangle of $\mathcal{T}$ having its basis on a boundary component of some annulus, and its tip on the other boundary component.

The Cannon–Thurston map. Next, I showed that the combinatorics of a veering triangulation are also related to the hyperbolic geometry of M via the *Cannon–Thurston map*, which we now define. Let D (a disk) be the universal cover of the fiber S. The inclusion $S \to M$ lifts to a map $\iota : D \to \mathbb{H}^3$ between the universal covers, which turns out to extend continuously to a boundary map $\bar{\iota} : \mathbb{S}^1 \to \mathbb{S}^2$. Cannon and Thurston [3] proved the surprising fact that $\bar{\iota}$ is a (continuous) *surjection* from the circle to the sphere. The endpoints of any leaf of $\lambda^\pm$ have the same image under $\bar{\iota}$, and this in fact generates all the identifications occurring under $\bar{\iota}$.

The connection with the veering triangulation and $\mathcal{T}$ is as follows. Choose an ideal vertex of the ideal triangulation of M; call it ∞. The hyperbolic metric gives a natural identification between $\mathbb{S}^2 - \{\infty\}$ and the universal cover of the toroidal link of ∞ in M. This universal cover (a plane Π) receives a topological triangulation $\widetilde{\mathcal{T}}$, lifting $\mathcal{T}$, in which the annuli of $\mathcal{T}$ become infinite vertical strips. (The vertices of $\widetilde{\mathcal{T}}$ are well-defined points with algebraic coordinates in $\mathbb{R}^2$, although higher skeleta of $\widetilde{\mathcal{T}}$ are only defined up to isotopy.) It turns out that the surjection $\bar{\iota} : \mathbb{S}^1 \to \Pi \cup \{\infty\}$ fills out Π by filling out in ordered succession a $\mathbb{Z}^2$-collection of topological disks, column by column, with columns being travelled alternately up and down. Each column corresponds to the interface A between adjacent infinite strips of $\widetilde{\mathcal{T}}$, and each topological disk δ corresponds to a basis $\beta \subset A$ of a triangle of $\widetilde{\mathcal{T}}$, with $\partial\beta \subset \partial\delta$. Two consecutive disks intersect at exactly one point, a vertex of $\widetilde{\mathcal{T}}$. Arbitrary disks intersect only (if at all) along their Jordan-curve boundaries, and the disks meet four at each vertex of $\widetilde{\mathcal{T}}$.

Although describing the full combinatorics requires a more elaborate dictionary between the foliations $\lambda^{\pm}$, the triangulation $\mathcal{T}$, and the Cannon–Thurston map $\bar{\iota}$, we can state the first entry of this dictionary as follows.

Theorem. *Suppose the hyperbolic* 3*-manifold M is a pseudo-Anosov mapping torus such that all singularities of the invariant foliations $\lambda^{\pm}$ occur at punctures of the fiber S. Let $\widetilde{\mathcal{T}}$ be the topological (doubly periodic) triangulation of the plane arising from the veering triangulation of M, and $\mathcal{D} = \{\delta_i\}_{i\in I}$ be the decomposition of the plane into topological disks arising from the Cannon–Thurston map. Then $\widetilde{\mathcal{T}}$ and $\mathcal{D}$ have the same vertex set.*

This connection was previously known for the punctured torus by work of Cannon and Dicks [2].

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Lens Space Recognition is in NP

Saul Schleimer

(joint work with Marc Lackenby)

We prove the following.

Theorem 3. The recognition problem for lens spaces lies in **NP**.

1. History

One way to begin the story of decision problems in three-manifolds is Haken's solution to the unknotting problem; he gave an algorithm [5] to decide if a polygonal loop in $\mathbb{R}^3$ is isotopic to the unknot. Haken introduced *normal surfaces*, a PL analogue of minimal surfaces. Jaco and Oertel [8], using Haken's techniques, gave an algorithm to decide if a closed, orientable, irreducible three-manifold contains a two-sided incompressible surface.

The three-sphere recognition problem was first shown to be decidable by Rubinstein [12], using *almost normal surfaces*, a PL analogue of index one minimal surfaces. His solution was improved upon by Thompson [15], using *thin position*: this is a mini-max principle (for knots in three-manifolds) due to Gabai [4]. Later, Casson [3] showed that the problem lies in **EXPTIME**. His proof introduced the concept of *crushing* triangulations along normal two-spheres. This reduces the number of tetrahedra monotonically and avoids the cutting procedure used in the earlier proofs.

Hass, Lagarias, and Pippenger [6] showed that the unknotting problem lies in **NP**; given an unknot of length n there is a normal spanning disk (provided by Haken) that can be checked in polynomial time. One delicate point is checking that the disk is connected. In the other direction, Agol, Hass, and Thurston [2] gave one of the first lower bounds, showing that determining the genus of a knot in an arbitrary manifold is **NP**-complete. Among other tools, they gave a polynomial-time algorithm to count the number of components of a normal surface. The second author [13] used their techniques, and a new polynomial-time algorithm to normalize almost normal surfaces, to show three-sphere recognition lies in **NP**. Ivanov [7] gave another solution, more in the spirit of [6].

Agol [1] announced a proof, using sutured manifold hierarchies, that the unknot recognition problem lies in co-**NP**. Very recently, Kuperberg [9] showed that the generalized Riemann hypothesis implies the same result, using several striking results from three-manifold topology, algebraic geometry, and complexity theory.

2. Background

Recall that the three-sphere is given by

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}.$$

The recognition problem for the three-sphere is the following. Given a triangulated three-manifold (M, Δ) determine if M is homeomorphic to the three-sphere.

Suppose that $p, q \in \mathbb{N}$ are given so that $1 \leq q \leq p$ and $\gcd(p, q) = 1$. Define $\zeta = \exp(2\pi i/p)$ and define

$$L(p, q) = S^3 \Big/ (z, w) \sim (\zeta z, \zeta^q w).$$

This is the *lens space* corresponding to the ordered pair (p, q). The recognition problem for lens spaces is the following. Given a triangulated three-manifold (M, Δ) determine if M is homeomorphic to a lens space.

Note for any fixed (p, q) the recognition problem for $L(p, q)$ reduces to the recognition problem for S^3. This follows from the following covering trick. To certify a three-manifold M is a lens space it suffices to give a cover $\rho\colon N \to M$ and to certify

- N is a three-sphere and
- the deck group $\mathrm{Deck}(\rho)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

However, this does not solve the lens space recognition problem stated above, because p may be exponentially large in terms of the given triangulation Δ.

Let $L_z = \{(z, w) \in S^3 \mid w = 0\}$ and $L_w = \{(z, w) \in S^3 \mid z = 0\}$ be the z and w–axes in S^3. Then $C = \{(z, w) \in S^3 : |z| = |w|\}$ is the *Clifford torus* separating L_z from L_w. Let V_z and V_w be the components of $S^3 - n(C)$ containing L_z and L_w, respectively. Then each of V_z and V_w are *solid tori*; they are homeomorphic to $S^1 \times D^2$. We say that the triple (C, V_z, V_W) is a *splitting* of S^3. If $\rho\colon S^3 \to L(p, q)$ is the standard covering map, then the images $T = \rho(C)$, $V = \rho(V_z)$, and $W = \rho(V_w)$

are again a torus and solid tori, respectively. Thus (T, V, W) is a splitting of $L(p, q)$. It is not hard to show that the lens spaces (and $S^1 \times S^2$) are the only three-manifolds admitting such splittings.

3. Almost normal splittings

Suppose that (M, Δ) is a triangulated three-manifold. For every tetrahedron $\tau \in \Delta$ we have a map $\pi_\tau \colon \tau \to M$ that is an embedding on interior(τ) and sends faces to faces. If $X \subset M$ then define $X_\tau = \pi_\tau^{-1}(X)$. A surface $S \subset M$, transverse to the skeleta of Δ, is *normal* if for every tetrahedron τ all components of S_τ are *normal disks*. See the left-hand side of Figure 1.

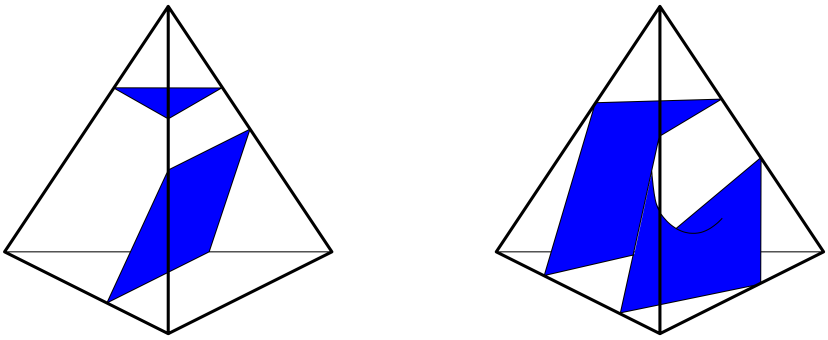

Figure 1. Two of the seven normal disks are shown on the left. The almost normal octagon is shown on the right.

The surface S is *almost normal* if all components of all preimages are normal disks, except for perhaps one component which is either an *almost normal octagon* (shown in Figure 1, on the right) or an *almost normal annulus*. The latter is obtained by connecting a pair of normal disks in τ by a tube parallel to an arc of the one-skeleton $\tau^{(1)}$. We can now state a result of Rubinstein and of Stocking.

Theorem 1. [11, 14] *Suppose that $p \geq 2$. For any triangulation Δ of $L(p, q)$ the splitting torus T can be isotoped to be almost normal.* $\square$

4. Short core curves

Suppose that $V = S^1 \times D^2$ is a solid torus. Any simple closed curve $\alpha \subset V$, isotopic to $S^1 \times \{\text{pt}\}$, is called a *core curve*. Suppose that Δ is a triangulation of V. We say α is *straight* with respect to Δ if

- α is transverse to the skeleta of Δ and
- for every tetrahedron $\tau \in \Delta$, the preimage α_τ is a collection of Euclidean arcs.

We can now state a result of the first author.

Theorem 2. [10] *Suppose that Δ is a triangulation of $V = S^1 \times D^2$. Then there is a straight core curve $\alpha \subset V$ so that $|\alpha_\tau| \leq 18$, for every tetrahedron $\tau \in \Delta$.* $\square$

5. Recognizing lens spaces

We are now equipped to state our main theorem.

Theorem 3. *The recognition problem for lens spaces lies in* **NP***.*

Proof sketch. Suppose (M, Δ) is a triangulated lens space, other than S^3. Let (T, V, W) be an almost normal splitting of M. Note that V inherits a cell structure, by gluing together the three-cells of V_τ, as τ ranges over the tetrahedra in Δ.

Unfortunately we do not have a good upper bound on the number of three-cells in V. Instead we find a much smaller cell structure on V by amalgamating the *parallel pieces*: the components of V_τ cobounded by normal disks of T_τ of the same type. The remaining pieces of V_τ are called *core pieces.* Each such union is called a *parallelity bundle* because it naturally has the structure of an orientable I–bundle over a surface (typically with boundary). There are at most a linear number (in $|\Delta|$) of core pieces in V and, similarly, of parallelity bundles in V.

After a further amalgamation step we arrange matters so that all but perhaps one of the parallelity bundles are trivial I–bundles over disks and annuli. The final bundle, if it exists, is an orientation I–bundle over a Möbius band. If this appears we call it a *central* bundle. There are certain subtleties that arise when both V and W have central bundles. Modulo such details, we next use a variant of Theorem 2 to obtain a core curve $\alpha \subset V$ that spends at most a linear amount of length in the parallelity bundles. Thus the total length of α is at most linear in $|\Delta|$.

Now, given α, let $V' = V - n(\alpha)$. Thus $V' \cong I \times T$ and so $M' = M - n(\alpha) = W \cup V' \cong S^1 \times D^2$ has a linear-sized triangulation.

So, to prove Theorem 3 it suffices to present α and then certify that M' is a solid torus by, say, appealing to the results of [13]. $\square$

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The Complexity of Recognizing the 3-Sphere

Joel Hass
(joint work with Greg Kuperberg)

Algorithms for topological problems in 3-manifold theory go back at least to Dehn. The analysis of the running times or computational complexity of such algorithms is more recent. Some early results in this direction are in [3], which analyzed the complexity of problems such as determining if a curve is unknotted, or more generally determining its genus. In particular, it was shown in [3] that UNKNOTTING, the problem of determining whether a curve in the 3-sphere represents the trivial knot, lies in the complexity class NP. Very recently Greg Kuperberg established that UNKNOTTING also lies in coNP, assuming the Generalized Riemann Hypothesis. A major remaining question is whether problems such as UNKNOTTING have an algorithm with polynomial running time, or whether they are NP-Complete.

In recent work with Greg Kuperberg we have shown that given a triangulated 3-manifold, the problem of recognizing whether it is homeomorphic to the 3-sphere lies in the complexity class NP intersect coNP, assuming the Generalized Riemann Hypothesis.

Specifically, we address the problem

Problem: 3-SPHERE RECOGNITION
Instance: A triangulated 3-dimensional manifold M.
Question: Is M homeomorphic to the 3-sphere?

An algorithm that recognizes the 3-sphere was given by Rubinstein [10] and by Thompson [12]. Casson gave an exponential upper bound on the running time of such an algorithm, and Schleimer [11] showed that it lies in NP.

We prove the following.

Theorem 3-SPHERE RECOGNITION is in the complexity class **coNP**, assuming the Generalized Riemann Hypothesis.

The class of problems that lie in both NP and coNP, but are not known to be polynomial, is fairly limited. These results can be viewed as presenting some evidence that problems such as UNKNOTTING and 3-SPHERE RECOGNITION are polynomial. The results serve as stronger evidence that they are not NP-complete, as if they were then certain widely believed conjectures in complexity theory or number theory are false.

The proof proceeds by pursuing different strategies depending on the identity of$M$. Given that M is not a 3-sphere, it falls into one of the following (overlapping) classes:

(1) $H_1(M; Z) \neq 1$,
(2) $\pi_1(M)$ is finite,
(3) M is a Seifert fibered space,
(4) M is hyperbolic,
(5) M contains an incompressible torus,
(6) M is a connect edsum.

In each case, we certify that M is not the 3-sphere in polynomial time, either by giving a non-trivial homomorphism from $\pi_1(M)$ to a finite group or by showing that $\pi_1(M)$ has a subgroup isomorphic to $Z \oplus Z$. As with Kuperberg's argument in [8], our methods are based on those of [7] and [9]. An application of techniques in [1] and the 3-manifold structure results of [6] and [4] allow for the polynomial time certification of $Z \oplus Z$ subgroups.

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Face Enumeration on Manifolds

Ed Swartz

This talk was a survey of results over the last few years involving the face numbers of triangulations of manifolds. We consider two types of triangulations: simplicial complexes and semi-simplicial complexes. Like simplicial complexes, the closed cells of semi-simplicial complexes are combinatorially simplices, however, there may be more than one cell for a given set of vertices. We indicate the first by using a subscript c, as in Δ_c. To indicate a semi-simplicial complex we use a subscript ss as in Δ_{ss}. If there is no subscript, then our statement applies to both types of complexes.

Throughout we will assume that Δ is a connected $(d-1)$-dimensional manifold with or without boundary. In either case the main combinatorial invariant under consideration is the f-vector of Δ, $(f_{-1}, f_0, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces. In particular f_{-1} is always one representing the empty face. It turns out that it is easier to study a linear transformation of the f-vector known as the h-vector. The h-vector is $(h_0, h_, \ldots, h_d)$ and is defined so that the following generating equation holds.

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum h_i t^{d-i}.$$

From this it is easy to see that $(f_{-1}, \ldots, f_{i-1})$ determines $(h_0, \ldots, h_i)$ and vice versa. Some simple to check formulas are

$$h_0 = 1, \quad h_1 = f_0 - d, \quad h_2 = f_1 - \binom{d-1}{1} f_0 + \binom{d}{2}, \quad h_d = (-1)^{d-1}\widetilde{\chi}(\Delta),$$

where $\widetilde{\chi}(\Delta)$ is the reduced Euler characteristic of the complex.

The first motivation for introducing the h-vector is the following generalization of the Dehn-Sommerville relations for manifolds due to Klee.

Theorem 1. [4] *If Δ is a manifold without boundary, then*

$$h_{d-i} - h_i = (-1)^i \binom{d}{i} [\chi(\Delta) - \chi(S^{d-1})].$$

An amusing consequence of the result is that if d is even, setting i to $d/2$ proves that the Euler characteristic must be zero. It has been a folk theorem for over 40 years that these are the only linear relations among the f_i. Since knowledge of $h_0, \ldots, h_{\lceil d/2 \rceil}$ determines the entire h-vector (and hence the f-vector), it has become common to encode the information in the g-vector which is given by

$$g_i = h_i - h_{i-1} \quad \text{for} \quad 0 \le i \le d/2.$$

Inequalities have been much harder to come by. Most recent results involve a heavy dose of commutative algebra. One of the few results before the introduction of this tool is due to Brehm and Kühnel.

Theorem 2. [2] *Let Δ_c be a PL-manifold. If $\pi_1(\Delta_c)$ is not trivial and $d \geq 4$, then $f_0 \geq 2d+1$.*

This theorem is optimal as for every $d \geq 4$ there is a nonsimply-connected PL-manifold Δ_c with exactly $2d+1$ vertices. Later, Bagchi and Datta showed that these complexes are the unique simplicial complexes with these properties [1].

In the 70's Richard Stanley, building on work of Hochster and Reisner, revolutionized the subject of face enumeration with the use of the face ring, also known as the Stanley-Reisner ring, for simplicial complexes. Later he introduced a similar algebraic gadget for semi-simplicial complexes [12]. All of the following upper and lower bounds depend on these tools.

1. Upper bounds

Let $C(n,d)$ be the cyclic d-polytope with n vertices, $n \geq d+1$. While we only state the following for odd-dimensional manifolds, it is known to hold for many even-dimensional manifolds.

Theorem 3. *(Novik)* [9] *If Δ_c is an odd-dimensional manifold without boundary, then for all i, $f_i(\Delta_c) \leq f_i(C(f_0(\Delta_c),d))$.*

A sequence $(a_0,\dots,a_d)$ of nonnegative integers is an *M-sequence* if it is the degree sequence of an order ideal of monomials. See [11, Theorem II.2.2] for a nonlinear arithmetic characterization of M-vectors.

Theorem 4. *(S. '12) If Δ_c is a closed manifold with $d \geq 6$, then*

$$(1, g_1, g_2 - \binom{d+1}{2}\beta_1,\ \max[0, g_3 + \binom{d+1}{3}\beta_1])$$

is an M-vector.

2. Lower bounds

Theorem 5. *(Novik - S. '09)* [10] *Let Δ_c be a manifold without boundary with $d \geq 4$. Then $f_1 - df_0 + \binom{d+1}{2} = g_2 = h_2 - h_1 \geq \binom{d+1}{2}\beta_1$, where $\beta_1 = \dim_k H_1(\Delta_c;k)$.*

For semi-simplicial complexes more is known. In order to state the results, define h''_i for $0 < i < d$ by

$$h''_i(\Delta) = h_i(\Delta) + \binom{d}{i}\left[-\beta_{i-1} + \beta_{i-2} - \cdots + (-1)^{i-1}\beta_1\right].$$

Theorem 6. *(Novik - S. '09)* [10] *Let Δ_{ss} be a manifold without boundary and $d \geq 4$. Then*

$$h''_i \geq 0.$$

This result, a parity result similar to Masuda's proof of Stanley's conjectured characterization of h-vectors of semi-simplicial complexes homeomorphic to spheres [6, 12], and some imaginative constructions have allowed Murai to give complete characterizations of h-vectors of semi-simplicial complexes homeomorphic to products of spheres [8].

3. Complexity

Now suppose M is a manifold with boundary with $d \geq 4$. Define

$$\Gamma(M) = \min_{|\Delta_c|=M} h_2 - h_{d-1} - d \cdot h_d = \min_{|\Delta_c|=M} h_2 - \text{ \# interior vertices.}$$

Kalai proved that $\Gamma(M) \geq 0$ and equals zero if and only if M is a ball [3].

Theorem 7. *(S. '11) Fix d and $G \geq 0$. Then the number of PL-homeomorphism types of $(d-1)$-dimensional M with $\Gamma(M) \leq G$ is finite.*

As in Matveev complexity [7], for closed M, define $\Gamma(M) = \Gamma(M - B^d)$. For closed manifolds, the above theorem is a generalization of [13]. It was observed in [5] that there are positive constants A, B such that for closed irreducible three-manifolds M not equal to $\mathbb{R}P^3, S^2 \times S^1$ or $L(3,1)$,

$$c(M) \leq A \cdot \Gamma(M) \quad \text{and} \quad \Gamma(M) \leq B \cdot c(M),$$

where $c(M)$ is the Matveev complexity of M. How far does the analogy go? Is $\Gamma(M_1 \# M_2) = \Gamma(M_1) + \Gamma(M_2)$?

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On the Generalized Lower Bound Conjecture for Polytopes and Spheres

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(joint work with Satoshi Murai)

Background and results The study of face numbers of polytopes is a classical problem. For a simplicial d-polytope P let $f_i(P)$ denote the number of its i-dimensional faces, where $-1 \leq i \leq d-1$ ($f_{-1}(P) = 1$ for the emptyset). The numbers $f_i(P)$ are conveniently described by the *h-numbers*, defined by

$$h_i(P) = \sum_{j=0}^{i} (-1)^{j-i} \binom{d-j}{i-j} f_{j-1} \quad \text{for} \quad 0 \le i \le d.$$

The Dehn-Sommerville relations assert that $h_i(P) = h_{d-i}(P)$ for all $0 \le i \le \lfloor \frac{d}{2} \rfloor$, generalizing the Euler-Poincaré formula.

In 1971, McMullen and Walkup [11] posed the following *generalized lower bound conjecture* (GLBC), generalizing Barnette's *lower bound theorem* (LBT) [2, 3].

Conjecture 1. *(McMullen–Walkup) Let P be a simplicial d-polytope. Then*

- *(a)* $1 = h_0(P) \le h_1(P) \le \cdots \le h_{\lfloor \frac{d}{2} \rfloor}(P)$.
- *(b) For an integer $1 \le r \le \frac{d}{2}$, the following are equivalent:*
  - *(i)* $h_{r-1}(P) = h_r(P)$.
  - *(ii) P is $(r-1)$-stacked, namely, there is a triangulation K of P all of whose faces of dimension at most $d-r$ are faces of P.*

Around 1980 the *g-theorem* was proved, giving a complete characterization of the face numbers of simplicial polytopes. It was conjectured by McMullen[10], sufficiency of the conditions was proved by Billera–Lee [4] and necessity by Stanley [14]. Stanley's result establishes part (a) of the GLBC, using the hard Lefschetz theorem for projective toric varieties.

As for part (b), the implication (ii) $\Rightarrow$ (i) was shown in [11]. The implication (i) $\Rightarrow$ (ii) is easy for $r = 1$, and was proved for $r = 2$ as part of the LBT [2]. Our main goal is to sketch a proof of the remaining open part of the GLBC. Further, we generalize it to homology spheres admitting the weak Lefschetz property (WLP, to be defined later).

McMullen [9] proved that, to study Conjecture 1(b), it is enough to consider combinatorial triangulations. Thus we write a statement in terms of simplicial complexes. For a simplicial complex Δ on the vertex set V and a positive integer i, let

$$\Delta(i) := \{F \subseteq V : \mathrm{skel_i}(2^{\mathrm{F}}) \subseteq \Delta\},$$

where $\mathrm{skel_i}(2^{\mathrm{F}})$ is the i-skeleton of the simplex defined by F, namely the collection of all subsets of F of size at most $i + 1$.

For a simplicial d-polytope P with boundary complex Δ, we say that a simplicial complex K is a *triangulation* of P if its geometric realization is homeomorphic to a d-ball and its boundary is Δ. A triangulation K of P is *geometric* if in addition there is a geometric realization of K whose underlying space is P.

Theorem 1. *(Murai–N.) Let P be a simplicial d-polytope with the h-vector $(h_0, h_1, \ldots, h_d)$, Δ its boundary complex, and $1 \le r \le \frac{d}{2}$ an integer. If $h_{r-1} = h_r$ then $\Delta(d-r)$ is the unique geometric triangulation of P all of whose faces of dimension at most $d-r$ are faces of P.*

Note that the uniqueness of such a triangulation was proved by McMullen [9]. Moreover, it was shown by Bagchi and Datta [1] that if Conjecture 1(b) is true then the triangulation must be $\Delta(d-r)$.

Theorem 2. *(Murai–N.) Let* Δ *be a homology* $(d-1)$*-sphere having the WLP over a field* k *of characteristic* 0, $(h_0, h_1, \ldots, h_d)$ *the* h*-vector of* Δ*, and* $1 \leq r \leq \frac{d}{2}$ *an integer. If* $h_{r-1} = h_r$ *then* $\Delta(d-r)$ *is the unique homology* d*-ball with no interior faces of dimension at most* $d-r$ *and with boundary* Δ*.*

We say that Δ *has the WLP* over k if for generic linear forms $\theta_1, \cdots, \theta_d, \omega$ in the face ring $k[\Delta]$, $\Theta = (\theta_1, \cdots, \theta_d)$ is an l.s.o.p. and the multiplication maps $\times\omega : (k[\Delta]/(\Theta))_i \to (k[\Delta]/(\Theta))_{i+1}$ are either injective or surjective for any i (it follows they are injective for $i < d/2$ and surjective for $i \geq d/2$).

Note that an algebraic formulation of the g-conjecture (for homology spheres) asserts that any homology sphere has the WLP, see e.g. [15, Conjecture 4.22] for a stronger variation. If this conjecture holds, then Theorem 2 will extend to all homology spheres. Indeed, the case $r = 2$ in Theorem 2 was proved by Kalai [7], without the WLP assumption, as part of his generalization of the LBT to homology manifolds and beyond. Further, note that for $r \leq d/2$, if a homology $(d-1)$-sphere Δ satisfies that $\Delta(d-r)$ is a homology d-ball with boundary Δ, then Δ satisfies all the numerical conditions in the g-conjecture (including the nonlinear Macaulay inequalities), as was shown by Stanley [13].

Sketch of proof of Theorem 1: Denote $\Delta' = \Delta(d-r)$, and let conv(F) denote the convex hull of a subset F of $\mathbb{R}^d$.

Step 1: show that $\{\mathrm{conv(F)} : \mathrm{F} \in \Delta'\}$ is a geometric realization of Δ', denoted $||\Delta'||$. The argument is geometric, using Radon's theorem, and reminds results by McMullen [9] and Bagchi-Datta [1]. Thus, $||\Delta'|| \subseteq P$.

Step 2: assume by contradiction that $||\Delta'|| \neq P$. Use Alexander duality to show it implies $\tilde{H}_{d-1}(||\Delta'||; \mathbb{Q}) \neq 0$. Thus, by Reisner criterion we will be done by the next step:

Step 3: show that the face ring $\mathbb{Q}[\Delta']$ is Cohen-Macaulay (CM) of Krull dimension $d+1$ over $\mathbb{Q}$. The first thing to observe here is that the WLP of Δ implies $\Delta(d-r) = \Delta(r-1)$, as was shown by Kalai [8] and Nagel [12]. Thus, the face ideals satisfy $I_{\Delta'} = (I_\Delta)_{\leq r}$ = the ideal generated by the minimal generators of degree at most r in I_Δ. Other ingredients include passing to the generic initial ideal and using Green's *crystallization principle* [6].

Sketch of proof of Theorem 2:

Step 1: as before, show $k[\Delta']$ is CM of dimension $d+1$. Thus, by Reisner criterion, all homology groups of face links in Δ vanish except maybe in the dimension of the face link.

Step 2: by Hochster formula and the local duality theorem, for any $F \in \Delta'$, the face link satisfies $\dim_k \tilde{H}_{d-|F|}(\mathrm{lk}_{\Delta'}(\mathrm{F}); \mathrm{k}) = \dim_{\mathrm{k}}(\Omega_{\mathrm{k}[\Delta']})_{\mathrm{e_F}}$ where Ω denotes the canonical module, with $\mathbb{Z}^n$-grading (n is the number of vertices in Δ') and e_F is the sum of standard basis element $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with $i \in F$.

Step 3: show that $\Omega_{k[\Delta']} \cong I_\Delta/I_{\Delta'}$, using some defining properties of the canonical module. To conclude, observe that for $F \in \Delta'$,

$$\dim_k (I_\Delta/I_{\Delta'})_{e_F} = \{ {}^{1\ F \notin \Delta,}_{0\ F \in \Delta.}$$

Last step – Uniqueness: again, the proof uses Alexander duality, and is inspired by Dancis [5].

Concluding remarks: 1. What more can be said on $(r-1)$-stacked polytopes for $r > 1$? McMullen [9] speculated that they are *regular*, and Bagchi-Datta [1] pointed out that even if they are *shellable* is not known.

2. In Theorem 2, suppose that Δ is a topological sphere / PL-sphere. Does it follow that Δ' is a topological ball / PL-ball?

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Triangulations of n-Manifolds

J. Hyam Rubinstein
(joint work with Marcel Bökstedt, Craig D. Hodgson, Henry Segerman and Stephan Tillmann)

1. Introduction

This report concerns three different projects. The first one is with Hodgson, Segerman and Tillmann on the topic of essential triangulations. The second project with Tillmann is on triangulations, where all codimension two faces are of even order, which we will refer to as even triangulations. The final project with Bökstedt and Tillmann is on Cartan–Hadamard metrics on even triangulations satisfying lower bound conditions on the degrees of codimension two faces. Cartan–Hadamard metrics coming from even triangulations can be viewed as a generalisation of Gromov's CAT(0) structures on manifolds with cubings.

2. Essential triangulations

In the joint project with Hodgson, Segerman, Tillmann we studied *essential* triangulations. In the case of closed n-manifolds, such a triangulation is assumed to have a single vertex and all edges are therefore loops. The property of being *essential* is that any edge loop is non-contractible. For n-manifolds which are the interiors of compact manifolds with boundary, triangulations are assumed to be ideal, i.e. all the vertices are 'at infinity' so are deleted from the manifold. In the ideal case, add in a copy of the boundary to compactify the manifold. Edges are then arcs which have ends on one or two boundary components. An ideal triangulation is *essential* if no edge with ends on a single boundary component can be homotoped into this boundary component, keeping its ends on the boundary.

Next, a triangulation of a closed manifold is called *strongly essential* if it is essential as above and moreover, any two edge loops are not homotopic. In the ideal case, no two edge loops are homotopic keeping their ends on the boundary. We give a number of key constructions.

If a closed Riemannian n-manifold has non-positive curvature then it has an essential triangulation. If the manifold admits a hyperbolic metric, it has a strongly essential triangulation. Conversely any *straight* one-vertex triangulation of a closed manifold with a non-positively curved metric is strongly essential. (A straight triangulation has the property that all the edges of the simplexes are geodesics. For the special case of constant curvature zero or -1, *geometric* triangulations are defined by the property that all the faces of the simplices are totally geodesic).

In dimension 3, both closed and ideal hyperbolic triangulations are strongly essential. So this is an interesting property which is required to find a geometric hyperbolic triangulation.

It is conjectured that every hyperbolic 3-manifold with torus boundary components has a decomposition into positive volume ideal hyperbolic tetrahedra. Such geometric triangulations were introduced by Thurston [10].

Next, we show that a closed 3-manifold M with $H_1(M, Z_2) \neq 0$ admits an essential triangulation.

Finally, we show that Haken 3-manifolds admit strongly essential triangulations, which are 'dual' to hierarchies. A Haken 3-manifold is P^2-irreducible (i.e. has no embedded 2-sided projective planes and any embedded 2-sphere bounds a 3-ball) and has a hierarchy, i.e. a system of embedded surfaces which are all incompressible and boundary incompressible, which cut the manifold up into 3-cells. (An incompressible surface Σ satisfies $\pi_1(\Sigma) \to \pi_1(M)$ is one-to-one and a boundary incompressible surface Σ is incompressible and properly embedded so that $\pi_1(\Sigma, \partial\Sigma) \to \pi_1(M, \partial M)$ is one-to-one.)

Note that to form a one-vertex triangulation in the closed case or an ideal triangulation in the bounded case, a careful argument is required. With suitable homological conditions, such a triangulation can be found dual to the hierarchy. However in general, crushing of part of the triangulation in the sense of [6] is needed.

3. Even triangulations

In joint work with Stephan Tillmann, we have obtained sufficient conditions for a 3-manifold to admit a triangulation, where all the edges have even order. The most interesting cases are where there are either one or two vertices, for closed 3-manifolds. Similarly for the interior of compact 3-manifolds with tori boundary components, one or two such components is also the most interesting case. The reason is that it is elementary to build even triangulations for all 3-manifolds, with many vertices in the closed case or many boundary tori in the non-empty boundary case. These sufficient conditions are also shown to be very close to being necessary.

Such triangulations have interesting properties and many explicit even triangulations can be constructed for bundles, open book decompositions and one- and two-sided Heegaard splittings. In particular, there is a symmetry representation of the fundamental group of the manifold into the symmetry group on $n+1$ letters associated with an even triangulation of an n-manifold. The covering manifolds corresponding to the kernels of various induced representations corresponding to partitions of $n+1$ contain special embedded *normal* submanifolds with useful properties. (A normal submanifold meets each simplex in a collection of simple properly embedded $(n-1)$-cells). In dimension 3 these normal surfaces consist entirely of quadrilaterals, one for each tetrahedron. The surfaces give either one- or two-sided Heegaard splittings of the 3-manifold.

Even triangulations have been studied extensively in [5, 7, 8, 9, 11], where they are called *foldable* triangulations.

4. Triangulations admitting Cartan–Hadamard metrics

In the project with Bökstedt and Tillmann, singular Riemannian metrics of non-positive curvature are induced on triangulated closed n-manifolds. Such metrics are often called *Cartan–Hadamard* since the manifolds have the same property as the Cartan–Hadamard theorem, namely their universal coverings are homeomorphic to $\mathbb{R}^n$. See [2] for a general treatment of Cartan–Hadamard spaces.

In [3], Gromov pointed out the importance of cubical complexes and cubical decompositions of manifolds, giving Cartan–Hadamard metrics or locally CAT(0) structures. In [1], a number of constructions of cubical decompositions of 3-manifolds giving such metrics were given.

In our project, the key idea is to decompose a regular Euclidean n-cube into $n!$ congruent n-simplices. Each of these simplices, denoted Δ, has all edge lengths of the form $1, \sqrt{2}, \dots, \sqrt{n}$ and dihedral angles between codimension one faces equal to one of $\frac{\pi}{k}$, for $k = 2, 3, 4$.

It is straightforward to see that given a closed n-manifold M with an even triangulation $\mathcal{T}$, lifting to the covering space $\tilde{M}$ corresponding to the representation of the fundamental group into the symmetric group on n letters, the simplices of the lifted triangulation $\tilde{\mathcal{T}}$ can be given the structure of the Δ so that the faces all glue together isometrically. However in general there will be cone type singularities at the codimension two faces. However, with the simple restriction that the appropriate families of codimension two faces have degrees at least $4, 6, 8$ corresponding to the dihedral angles of Δ being $\frac{\pi}{k}$, for $k = 2, 3, 4$, the induced metric is Cartan–Hadamard.

This construction can be viewed as a generalisation of locally CAT(0) cubical structures on manifolds, since all such examples also have triangulations of the above type, by subdividing the cubes into copies of Δ. However there are many new examples which can be obtained by this technique. As one example, in [1] it is shown that any branched covering of the 3-sphere over the figure 8 knot so that all the components of the branch set have degrees at least 3 has a Cartan–Hadamard metric, using objects called flying saucers. It is very natural to divide these flying saucers into copies of Δ giving a triangulation with a Cartan–Hadamard metric. Note in [4] it is proved that the figure 8 knot is *universal*, i.e. every closed orientable 3-manifold is a branched covering of the 3-sphere over it.

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Simplicial Complex Models for Arrangement Complements

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(joint work with Pavle V. M. Blagojević)

1. Why do we care?

1.1. **Arrangements and configuration spaces.** The configuration space of n labeled distinct points on a manifold

$$F(X,n) := \{(x_1,\dots,x_n) \in X^n : x_i \neq x_j \text{ for } i<j\}$$

appears in diverse contexts in topology (providing, for example, embedding invariants and models for loop spaces), knot theory, and physics (KZ equation, renormalization); see e.g. Vassiliev [20] and Fadell & Husseini [8]. In particular, the space

$$F(\mathbb{R}^d,n) = \{(x_1,\dots,x_n) \in \mathbb{R}^{d\times n} : x_i \neq x_j \text{ for } i<j\}$$

has been studied in great detail. It is the complement of an arrangement of linear codimension d subspaces in $\mathbb{R}^d$ whose intersection lattice (with the customary ordering by reversed inclusion) is the partition lattice Π_n of rank $n-1$. The cohomology is free, with Poincaré polynomial $\prod_{i=1}^{n-1}(1+i\,t^{d-1})$; see e.g. Björner[4], Goresky & MacPherson [13, Part III].

In particular, for $d=2$ the space $F(\mathbb{C},n) = F(\mathbb{R}^2,n)$ appears in work by Arnol'd related to Hilbert's 13th problem, continued in papers by works by Fuks, Deligne, Orlik & Solomon, and many others. It is a key example for the theory of (complex) hyperplane arrangements; see e.g. Orlik & Terao [17].

The space $F(\mathbb{R}^d,n)$ is the complement of a codimension d subset in $\mathbb{R}^{d\times n}$, so in particular it provides a $(d-2)$-connected space on which the symmetric group $\mathfrak{S}_n$ acts freely. Thus the inclusions $F(\mathbb{R}^2,n) \subset F(\mathbb{R}^3,n) \subset \cdots$ can be used to compute the cohomology of $\mathfrak{S}_n$; see Giusti & Sinha [12] for recent work, which is based on the Fox–Neuwirth stratification [11] of the configuration spaces $F(\mathbb{R}^d,n)$.

1.2. **Cell complex models.** A problem by R. Nandakumar and N. Ramana Rao [16] asks whether every bounded convex set P in the plane can be divided into n convex pieces that have equal area and equal perimeter. In [18] the same authors prove this for $n=2^k$ in the case where P is a convex polygon. Blagojević, Bárány & Szűcs [2] established the problem for $n=3$.

Karasev [15] and Hubard & Aronov [14] observed that a positive solution for the problem would — via optimal transport (cf. Villani [21]) and generalized Voronoi diagrams (cf. Aurenhammer et al. [1]) — follow from the non-existence of an equivariant map

$$F(\mathbb{R}^2, n) \longrightarrow_{\mathfrak{S}_n} S(\{(y_1, \dots, y_n) \in \mathbb{R}^n : y_1 + \cdots + y_n = 0\}) \simeq S^{n-2}.$$

A d-dimensional and more general version of the problem, to partition any sufficiently continuous measure on $\mathbb{R}^d$ into n pieces of equal measure that also equalize $d-1$ further continuous functions, could be solved by the non-existence of

$$F(\mathbb{R}^d, n) \longrightarrow_{\mathfrak{S}_n} S(\{(y_1, \dots, y_n) \in \mathbb{R}^{(d-1)\times n} : y_1 + \cdots + y_n = 0\}) \simeq S^{(n-1)(d-1)-1}.$$

In the cited works by Karasev and Hubard & Aronov this is approached via (twisted) Euler class computations on the one-point compactification of $F(\mathbb{R}^d, n)$ (which is not a manifold), which leads to the non-existence of these maps if n is a prime power.

Here we report about an alternative approach, via Equivariant Obstruction Theory (as developed by tom Dieck [7, Sect. II.3]). For this we need an equivariant cell complex model for $F(\mathbb{R}^d, n)$.

2. A method

We rely on a method developed in Björner & Ziegler [5] to obtain a compact cell complex model for the complements of linear subspace arrangements. For this let $\mathcal{A}$ be a finite arrangement of linear subspaces in a real vector space $\mathbb{R}^N$. Each k-dimensional subspace F of the arrangement is embedded into a complete flag of linear subspaces $F = F_k \subset F_{k+1} \subset \cdots \subset \mathbb{R}^N$. The union of these flags yields a stratification of $\mathbb{R}^N$ into relative-open convex cones. These cones are not usually pointed, but their faces are unions of strata. The barycentric subdivision of the stratification yields a triangulation of a star-shaped neighborhood of the origin in $\mathbb{R}^N$, which as a subcomplex contains a triangulation of the link of the arrangement. Its Alexander dual is the barycentric subdivision of a regular CW complex, realized as a geometric simplicial complex that is a *strong deformation retract of the complement.* Moreover, if the arrangement has a symmetry, and the flags are chosen to be compatible with the symmetry, then the resulting complex carries the symmetry of the arrangement.

3. Examples

Implementing the construction from [5] yields the Fox–Neuwirth stratification on the complement $F(\mathbb{R}^d, n)$ of the arrangement, but indeed our construction and proof uses the stratification on the full ambient space $\mathbb{R}^{d\times n}$.

Theorem 1. *There is a regular cell complex $\mathcal{F}(d, n)$ of dimension $(n-1)(d-1)$ that has $n!$ vertices and $n!$ facets (maximal cells), with a free cellular action of the symmetric group $\mathfrak{S}_n$ that is transitive on the vertices as well as on the facets.*

The barycentric subdivision $\operatorname{sd}\mathcal{F}(d, n)$ has a geometric realization in $F(\mathbb{R}^d, n)$ as an equivariant strong deformation retract.

Based on this model, our Equivariant Obstruction Theory calculation gives a complete answer to the equivariant map problem, and thus a simple combinatorial proof for the prime power case of the Nandakumar & Ramana Rao conjecture:

Theorem 2 ([6]). *An equivariant continuous map*

$$F(\mathbb{R}^d, n) \longrightarrow_{\mathfrak{S}_n} S(\{(y_1, \dots, y_n) \in \mathbb{R}^{(d-1)\times n} : y_1+\cdots+y_n = 0\}) \simeq S^{(n-1)(d-1)-1}.$$

does not exist if and only if n is a prime power.

At the combinatorial core of our calculation lies the fact, apparently first proved by B. Ram in 1909 [19], that $\gcd\{\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}\}$ equals p for any prime power $n = p^k$, and equals 1 otherwise.

4. Further Examples

In view of further applications to geometric measure partition problems, there is interest in constructing and analyzing cell complex models for spaces such as $F(S^d, n)$ (see Feichtner & Ziegler [9] and Basabe et al. [3]) as well as $F_{\pm}(S^d, n)$ (see [10]).

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The Diameter of Polytopes and Abstractions

Nicolai Hähnle

1. The polynomial Hirsch conjecture

Let $V \subseteq \binom{[n]}{d}$ be a set system with an adjacency structure given by an undirected graph $G = (V, E)$ such that the induced subgraph $G[star(F, V)]$ is connected for every $F \subseteq [n] = \{1, \dots, n\}$. As usual, we let $star(F, V) = \{H \in V \ : \ H \supseteq F\}$. This definition generalizes strongly connected pure $(d-1)$-dimensional simplicial complexes, which have a natural adjacency structure given by their dual graph. Our definition allows more freedom in the definition of adjacency.

We use this combinatorial structure to study the *polynomial Hirsch conjecture*, which is motivated by the question of computational complexity of the Simplex Method for linear programming. The conjecture claims that the diameter of the vertex-edge graph of every d-dimensional polytope (or polyhedron) with n facets is bounded by a polynomial in d and n. We can restrict the question to simple polytopes without loss of generality.

From a polar perspective, the question becomes the following. Given a simplicial d-dimensional polytope P with n vertices, can we give bounds on the diameter of its facet-ridge graph in terms of n and d? The boundary of P is a strongly connected $(d-1)$-dimensional simplicial complex, and so it is natural to ask the same question in the more general combinatorial setting defined initially.

The best known upper bound for the diameter of polytopes is $n^{1+\log d}$ [5], where $\log d$ is the base-2 logarithm. Furthermore, we know that the diameter of polytopes is linear in fixed dimension, in fact $diam(P) \leq \frac{1}{3}2^{d-2}(n-d+\frac{5}{2})$ for $d \geq 3$ [1, 7]. We show that these results also hold in the combinatorial setting we define, with one caveat: the linear upper bound becomes $2^{d-1}n$. The difference is easily explained: the proof uses induction on the dimension, and an upper bound of $\frac{2}{3}n$ for 3-dimensional polytopes follows from geometric arguments that do not apply in the combinatorial setting.

On the other hand, we construct a family of purely combinatorial examples with $d = n/4$ and diameter $\Omega(n^2/\log n)$ using Lovász' Local Lemma [4]. This is in contrast to the best known geometric constructions, whose diameter is only linear. In particular, Santos recently found a family of polytopes with diameter $(1 + \varepsilon)n$ for a small constant ε in sufficiently high but fixed dimension [9]. A (non-polytopal) family of simplicial complexes with slightly larger but still linear diameter can be constructed based on the Mani–Walkup sphere [8]. Whether in the geometric or the abstract combinatorial case, the gap between constructions and upper bounds is huge. See the surveys [6] for additional background on the Hirsch conjecture and [3] for the wider perspective of linear optimization.

2. Connected layer families

For the purpose of both upper bound proofs and combinatorial constructions, it is convenient to fix an $x_0 \in V$ and label all $v \in V$ by their distance to x_0. This suggests the definition of *connected layer families* as partitions of a set $V \subseteq \binom{[n]}{d}$ into *layers* $L_0, \ldots, L_\ell$ such that for every $i < j < k$ and for every $v \in L_i$ and $w \in L_k$ there exists a $u \in L_j$ such that $u \supset v \cap w$. That is, if some set of less than d *symbols* appears in layers L_i and L_k, then it must also appear on every layer in between. We say that ℓ is the diameter of the connected layer family. One easily shows that this definition is equivalent to the initial definition as far as the diameter question is concerned. Let $f(d, n)$ denote the maximum diameter of a connected layer family with parameters d and n. The key results mentioned in the previous section can be summarized as $f(d, n) \leq \min\{n^{1+\log d}, 2^{d-1}n\}$ and $f(n/4, n) \geq \Omega(n^2/\log n)$.

Let us generalize our definition to allow multisets of $[n]$ and denote by $\tilde{f}(d, n)$ the maximum diameter given this further relaxation. One can easily show that $f(d, n) \leq \tilde{f}(d, n) \leq f(d, dn)$. The first inequality is trivial, and for the second inequality we replace multi-subsets of X by subsets of $\{x_i \ : \ x \in X, 1 \leq i \leq d\}$ by replacing e.g. the 5-element multiset $xxyyy$ by the 5-element set $x_1x_2y_1y_2y_3$ (we omit braces from the notation to reduce visual noise). Hence the two definitions are polynomially equivalent in the sense that if $f(d, n)$ is polynomially bounded then so is $\tilde{f}(d, n)$, and vice versa.

In the multiset setting, there is an easy construction to obtain the inequality $\tilde{f}(d, n) \geq d(n - 1)$, e.g. for $d = 3$:

$$000 - 001 - 011 - 111 - 112 - 122 - 222 - 223 - 233 - 333 - \ldots$$

We will sketch proofs that $d(n - 1)$ is in fact an upper bound on the diameter of connected layer families if we require certain additional extremal properties.

The first case is when every layer contains only a single element $v \in V$, as in the example given above. We use a "non-revisiting" argument. Fix a layer L_i. Since every layer contains only one multiset, there is some symbol x that appears k-fold on layer L_i, but strictly less than k-fold on layer L_{i+1}. By the definition of connected layer families, the symbol x cannot appear k-fold on any layer L_j with $j > i$, that is, x cannot be "revisited" with multiplicity k.

The second case is when V is complete in the sense that it contains every possible d-element multiset that can be built from the n symbols available. We use the fact that for every multiset F, the collection of multisets containing F induces a connected layer family with the same layer structure but of dimension $d-|F|$. This allows us to proceed by induction. Take some $u \in L_0$ and $v \in L_\ell$. Take any $x \in u$ and any $y \in v$. Then by completeness, the multiset $v \setminus \{y\} \cup \{x\}$ must be present on some layer L_j, and induction on the dimension yields $j \leq \tilde{f}(d-1, n) \leq (d-1)(n-1)$ and $\ell - j \leq \tilde{f}(1, n) \leq (n-1)$, hence $\ell \leq d(n-1)$.

Note that a diameter of $d(n-1)$ can also be achieved when V is complete. Use symbols $\{0, 1, \ldots, n-1\}$ and label every d-element multiset by the sum of its elements. It is easy to verify that the layer structure indicated by this labelling satisfies the definition of a connected layer family.

3. Questions and a Conjecture

The main open question is of course the polynomial Hirsch conjecture. Furthermore, the fact that $d(n-1)$ is tight for the diameter in these extremal cases prompts us to *conjecture* that $\tilde{f}(d, n) = d(n-1)$. This is known to be true for $d \leq 2$ and $n \leq 3$. We have also verified the conjecture for additional small values of d and n on a computer using state of the art satisfiability solvers.

Note, however, that when we further relax the definition and replace multisets by ordered tuples, there exists a connected layer family with $d = 3$, $n = 6$ and diameter 16. This example was found by a satisfiability solver.

One logical next step would be trying to understand the case $d = 3$ better. In the abstract setting, we know that $3(n-1) \leq \tilde{f}(3, n) \leq 4n - c$, where the second inequality follows from a Barnette–Larman style argument. Can our conjecture be proved for this special case, or can we find a counter-example?

What if we restrict to simplicial complexes that arise as triangulations of a surface? Using Barnette–Larman style arguments with an appropriate base case for the induction, we can show upper bounds of $2n$ and n for surfaces with and without boundary, respectively. Are these upper bounds tight? Recall that an upper bound of $\frac{2}{3}n$ holds for the boundary of a 3-dimensional polytope, so there is room for improvement or, possibly, the construction of non-polytopal surfaces with higher diameter.

We also mention the recent joint work with Bonifas et al. [2] in which we bound the diameter of polyhedra by a function depending on the subdeterminants of the coefficient matrix A of the system $Ax \leq b$ defining the polyhedron. This suggests a different approach to the diameter question by bounding the diameter of polyhedra in terms of other parameters than just n and d.

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Expansion properties of random simplicial complexes

Matthew Kahle

1. Random graphs and simplicial complexes

1.1. **Random graphs.** Define $G(n,p)$ to be the probability space of graphs on vertex set $[n] = \{1, 2, \ldots, n\}$, where each edge has probability p, jointly independently. An important early theorem in probabilistic topology is the following result of Erdős and Rényi.

Theorem 1 (Erdős–Rényi, 1959)**.** *Let $\epsilon > 0$ be fixed, and $G \in G(n,p)$.*

(1) If

$$p \geq \frac{(1+\epsilon)\log n}{n},$$

then

$$\mathbf{Pr}[G \text{ is connected}] \to 1,$$

(2) and if

$$p \leq \frac{(1-\epsilon)\log n}{n},$$

then

$$\mathbf{Pr}[G \text{ is connected}] \to 0,$$

as $n \to \infty$.

There are now several results for random simplicial complexes which provide higher-dimensional analogues of Theorem 1. A few of these are briefly surveyed in the following.

1.2. **Random 2-complexes.** Define $Y(n,p)$ to be the probability space of 2-dimensional simplicial complexes with vertex set $[n]$, edge set $\binom{[n]}{2}$, and such that every 2-dimensional face has probability p, jointly independently.

Linial and Meshulam provided a cohomological analogue of Theorem 1.

Theorem 2. *(Linial–Meshulam, 2006) Let $\epsilon > 0$ be fixed and $Y \in Y(n,p)$. Then*

$$\mathbf{Pr}[H^1(Y, \mathbb{Z}/2) = 0] \to \begin{cases} 1 & : \ p \geq (2+\epsilon)\log n/n, \\ 0 & : \ p \leq (2-\epsilon)\log n/n. \end{cases}$$

A group G is said to have *Property (T)* if the trivial representation of G is an isolated point in its unitary dual equipped with the Fell topology. Informally, G has (T) if whenever it acts unitarily on a Hilbert space and has almost invariant vectors, it has a nonzero invariant vector.

Theorem 3. *(Hoffman–K.–Paquette, 2012) Let $\epsilon > 0$ be fixed and $Y \in Y(n,p)$. Then*

$$\mathbf{Pr}[\pi_1(Y) \text{ has property (T)}] \to \begin{cases} 1 & : \ p \geq (2+\epsilon)\log n/n, \\ 0 & : \ p \leq (2-\epsilon)\log n/n. \end{cases}$$

Both Theorem 3 and 2 imply that the threshold for vanishing of $H^1(Y, \mathbb{Q})$ is $2\log n/n$.

Our main tool is the following theorem of Żuk.

Theorem 4. *(Żuk) If Δ is a finite, connected, pure 2-dimensional simplicial complex, such that for every vertex v, the link $lk_\Delta(v)$ is connected and has spectral gap of normalized Laplacian satisfying $\lambda_2[lk_\Delta(v)] > 1/2$, then $\pi_1(\Delta)$ has property (T).*

We also require new results for the spectral gap of $G \in G(n,p)$, as in the following.

Theorem 5. *(Hoffman–K.–Paquette, 2012) Fix $k \geq 0$ and $\epsilon > 0$, and let $G \in G(n,p)$. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ be the eigenvalues of the normalized Laplacian of G. There is a constant $C = C(k)$ so that when*

$$p \geq \frac{(k+1)\log n + C\sqrt{\log n}\log\log n}{n}$$

is satisfied, then

$$\lambda_2 > 1 - \epsilon,$$

with probability at least $1 - o(n^{-k})$.

1.3. **Random flag complexes.** A different sort of random simplicial complex is the random flag complex. Let $G \in G(n,p)$ be an Erdős-Rényi random graph (on n vertices with each edge having probability p, independently). Let $X \in X(n,p)$ be the clique complex (or flag complex) of $G \in G(n,p)$, i.e. the maximal simplicial complex compatible with G.

It should be emphasized that every simplicial complex is homeomorphic to a flag complex, e.g. by barycentric subdivision, so $X(n,p)$ puts a measure on a wide range of topologies.

Topological properties of this kind of random simplicial complex were studied earlier in [2]. The new result in [3] is the following. Note that the Erdős–Rényi Theorem corresponds to $k = 0$.

Theorem 6. *(K., 2012) Let $k \geq 1$ and $\epsilon > 0$ be fixed, and $X \in X(n,p)$.*

(1) If

$$p \geq \left(\frac{(k/2 + 1 + \epsilon) \log n}{n} \right)^{1/(k+1)},$$

then $\mathbf{Pr}[H^k(X, \mathbb{Q}) = 0] \to 1$,

(2) and if

$$n^{-1/k+\epsilon} \leq p \leq \left(\frac{(k/2 + 1 - \epsilon) \log n}{n} \right)^{1/(k+1)},$$

then $\mathbf{Pr}[H^k(X, \mathbb{Q}) = 0] \to 0$, as $n \to \infty$.

In proving this, the main tool is the following theorem of Ballman–Światkowski [1]. It is illustrative to compare Theorem 7 to Theorem 4.

Theorem 7. *(Garland, 1973; Ballman–Światkowski, 1997) If Δ is a pure k-dimensional simplicial complex, such that the link $lk_\Delta(\sigma)$ of every $(k-2)$-face σ is connected and has spectral gap satisfying*

$$\lambda_2[lk_\Delta(\sigma)] > 1 - 1/k,$$

then $H^{k-1}(\Delta, \mathbb{Q}) = 0$.

Applying universal coefficients, and together with several earlier results on random flag complexes, we have the following.

Corollary 1. *Fix $d \geq 0$, and let $X \in X(n,p)$ be a random flag complex, where*

$$n^{-2/d} \ll p \ll n^{-2/(d+1)}.$$

Then w.h.p. X is d-dimensional and $\widetilde{H}_i(X, \mathbb{Q}) = 0$ unless $i = \lfloor d/2 \rfloor$.

I.e. almost all d-dimensional flag complexes have all their (reduced, rational) homology in middle degree.

2. Open problems

So far we don't know much about how to handle torsion in homology of random complexes. I conjecture however that torsion is vanishing w.h.p., for a fairly wide range of p.

Conjecture 1. *If $X \in X(n,p)$ is a random flag complex with*

$$n^{-2/d} \ll p \ll n^{-2/(d+1)},$$

and if $d \geq 6$, then w.h.p. X is homotopy equivalent to a wedge of $\lfloor d/2 \rfloor$-spheres.

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Stable Complexity and Simplicial Volume of Manifolds

Bruno Martelli

(joint work with Stefano Francaviglia, Roberto Frigerio)

As customary in low-dimensional topology we define a *triangulation* of a closed PL manifold M^n to be a realization of M by a simplicial face-pairing of a finite set of n-dimensional simplexes. Let the *complexity* $c(M)$ be the minimal number of simplices in a triangulation of M. Such a quantity is clearly submultiplicative with respect to finite coverings, that is we have

$$c(\widetilde{M}) \leq d \cdot c(M)$$

for every degree-d covering $\widetilde{M} \stackrel{d}{\to} M$ of closed manifolds. Following Milnor and Thurston [1] we may promote this quantity to a multiplicative one by defining

$$c_\infty(M) = \inf_{\widetilde{M} \stackrel{d}{\to} M} \left\{ \frac{c(\widetilde{M})}{d} \right\}.$$

We call this new quantity the *stable complexity* of M. Stable complexity is now multiplicative under finite coverings, that is we have $c_\infty(\widetilde{M}) = d \cdot c_\infty(M)$ for any degree-d covering $\widetilde{M} \stackrel{d}{\to} M$. A quantity which is multiplicative under finite covering was called a *characteristing number* by Milnor and Thurston [1]: two famous characteristic numbers are the *Euler characteristic* $\chi(M)$ and Gromov's *simplicial volume* $\|M\|$.

We briefly recall the definition of $\|M\|$. Every real homology group $H_k(X, \mathbb{R})$ of any topological space X is equipped with a semi-norm defined as follows:

$$\|\alpha\| = \inf \left\{ |\lambda_1| + \ldots + |\lambda_k| \;\middle|\; \alpha = \lambda_1\sigma_1 + \ldots + \lambda_k\sigma_k \right\}$$

where the infimum is taken on all the representations of the cycle $\alpha \in H_k(X, \mathbb{R})$ as a linear combination $\lambda_1\sigma_1 + \ldots + \lambda_k\sigma_k$ of singular simplexes with real coefficients $\lambda_i \in \mathbb{R}$. The simplicial volume $\|M\|$ of a closed oriented connected n-manifold M is defined as the norm of its fundamental class $[M] \in H_n(M, \mathbb{R})$, that is:

$$\|M\| = \|[M]\|.$$

We are interested in the relation between the two characteristic numbers $c_\infty(M)$ and $\|M\|$. Since every triangulation gives rise to a fundamental cycle, it is immediate to prove that

$$\|M\| \leq c_\infty(M).$$

It is natural to ask whether the two quantities are actually equal on some interesting manifolds. It is reasonable to look first at aspherical manifolds, and more precisely at hyperbolic manifolds. It turns out, slightly surprisingly, that the two equalities are distinct for higher dimensional manifolds.

Theorem 1. *In every dimension $n \geq 4$ there is a constant $C_n < 1$ such that $\|M\| \leq C_n c_\infty(M)$ for every closed hyperbolic n-manifold M.*

Sketch of the proof. The main idea is actually simple. On a hyperbolic manifold M there is a well-known equality

$$\mathrm{Vol}(M) = v_n \|M\|$$

where v_n is the largest volume of a simplex in hyperbolic space $\mathbb{H}^n$, realized precisely by the ideal regular n-simplex. It is easy to check that the dihedral angle of a hyperbolic regular ideal n-simplex does *not* divide 2π when $n \geq 4$, and this implies that it is not possible to construct a triangulation of M such that "most" of its simplexes straighten to "big" simplexes, whose volumes are close to v_n. Some accurate estimates then show that we need more than $C_n^{-1}\mathrm{Vol}(M)/v_n$ simplexes to triangulate M, for some $C_n < 1$ that depend only on the dimension n. □

In dimension 2 we obviously have $\|M\| = c_\infty(M)$ for any hyperbolic surface$M$. The question for hyperbolic 3-manifolds is as far as we know still open: we do not know a single closed hyperbolic M for which $c_\infty(M) > \|M\|$, and we do not know a single closed hyperbolic manifold M for which $c_\infty(M) = \|M\|$. By replacing triangulation with spines – and hence taking Matveev complexity $c(M)$, which is a nice invariant defined on any compact M (possibly with boundary) – we can however prove the following.

Theorem 2. *The stable complexity c_∞ on 3-manifolds is additive on connected sums and on the pieces of the JSJ decomposition.*

Note that the simplicial volume is also additive on connected sums and JSJ decompositions (although the proof of this fact is certainly non trivial). It is easy to prove that $c_\infty(M) = \|M\| = 0$ on any geometric non-hyperbolic (*i.e.* Seifert or Sol) manifold M: this implies in particular that $c_\infty(M) = \|M\| = 0$ on any graph manifold. The hyperbolic case is thus indeed the only interesting one.

We ask here a question that, if answered positively, would imply that $\|M\| = c_\infty(M)$ on a closed hyperbolic 3-manifold M.

Question 3. *Does M virtually cover S^3 branching over the figure-eight knot with arbitrarily high degree?*

We mean the following: does for any $L > 0$ exist a finite covering $\widetilde{M} \to M$ and a branched covering $p\colon \widetilde{M} \to S^3$ with branching locus the figure-eight knot K, such that the ramification index on *each* component of $p^{-1}(K)$ is greater than L?

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Delaunay Mesh Generation of Surfaces

Tamal K. Dey

In this talk we focus on how to compute digital representation of a smooth surface in three dimensions with Delaunay meshes. For a given point set, it is known that Delaunay triangulations optimize various geometric properties and lend itself to systematic theoretical analysis. It is mainly because of these two reasons, Delaunay triangulations and their dual Voronoi diagrams have found their use in producing meshes with theoretical guarantees about shape and approximation qualities. We consider two versions of the problem that occur in various applications: (i) surface reconstruction, and (ii) mesh generation.

(i) Surface Reconstruction: In this problem, the input is a finite point set $P \subset \Sigma$ sampled from a smooth compact surface $\Sigma \subset \mathbb{R}^3$ without any boundary. Notice that Σ is not available as input. The task is to compute a simplicial complex K whose vertices are in P and whose underlying space $|K| \subset \mathbb{R}^3$ has the same topology of Σ and a geometry that approximates Σ. Amenta and Bern [1] proposed the first algorithm that provably computes a Delaunay subcomplex which is a 2-manifold and approximates the geometry of Σ. In this pioneering work, they introduce the important concepts of local feature size, ε-samples for surfaces, and connected them with the concept of restricted Delaunay triangulations.

Definition 1. *Local feature size is a function $f : \Sigma \to \mathbb{R}$ where $f(x)$ is the Euclidean distance of x to the medial axis of Σ.*

Definition 2. *A point sample $P \subset \Sigma$ is called an ε-sample of Σ if every point $x \in \Sigma$ has a sample point $p \in P$ so that $\|p - x\| \leq \varepsilon f(x)$.*

Definition 3. *Let* $\operatorname{Del} P$ *denote the Delaunay triangulation of $P \subset \mathbb{R}^3$. The restricted Delaunay triangulation of P with respect to Σ, denoted* $\operatorname{Del} P|_\Sigma$*, is the subcomplex of* $\operatorname{Del} P$ *where a simplex $\sigma \in \operatorname{Del} P$ belongs to* $\operatorname{Del} P|_\Sigma$ *if and only if its dual Voronoi face intersects Σ.*

Amenta and Bern proved that if P is an ε-sample of Σ for a sufficiently small $\varepsilon < 1$, the underlying space of restricted Delaunay triangulation $\operatorname{Del} P|_\Sigma$ becomes homeomorphic to Σ. Since Σ is not given, one cannot compute $\operatorname{Del} P|_\Sigma$. The challenge is to compute a complex that mimics $\operatorname{Del} P|_\Sigma$. The crust algorithm of Amenta and Bern [1] outputs such a complex with the guarantee that it is a 2-manifold and is Hausdorff-close to Σ with respect to the sampling density. However, the algorithm requires two computations of Delaunay triangulations and the proof of homeomorphism between output surface and Σ was missing. In a subsequent work, Amenta, Choi, Dey, and Leekha [2] gave a modified algorithm called Cocone that outputs a Delaunay mesh which requires computing the Delaunay triangulation only once. Furthermore, the output is proved to be homeomorphic to the sampled surface. The precise statement is:

Theorem 4. *Let P be an ε-sample of a smooth, compact, boundary-less surface $\Sigma \subset \mathbb{R}^3$. A Delaunay subcomplex $K \subset \operatorname{Del} P$ with vertex set P can be computed in $O(|P|^2)$ time with the following properties for $\varepsilon \leq 0.05$:*

(1) *The underlying space $|K|$ is homeomorphic to Σ (actually, there is an ambient isotopy taking $|K|$ to Σ).*
(2) *Every point in $|K|$ has a point $x \in \Sigma$ so that $\|p-x\| \leq O(\varepsilon)f(x)$. Similarly, every point x in Σ has a point p in $|K|$ so that $\|p-x\| \leq O(\varepsilon)f(x)$.*
(3) *Each triangle $t \in K$ has a normal making an angle $O(\varepsilon)$ with the normal to the surface Σ at any of its vertices.*

The software Cocone based on the Cocone algorithm and its variants is publicly available from the author's web-page.

(ii) Mesh generation: Unlike surface reconstruction, in this problem, the input is a smooth surface Σ, and we are asked to sample Σ and connect these sample points to create a surface mesh approximating Σ. The surface Σ may be input as an implicit equation, or by some polygonal approximation.

The introduction of the sampling theory for surface reconstruction lent to the development of provable algorithms for meshing smooth surfaces and volumes bounded by them. Cheng, Dey, Edelsbrunner, and Sullivan [5] showed how the Delaunay refinement of Chew [8] can be used to sample a surface so that a topological condition called Topological Ball property(TBP) [9] is satisfied. If TBP is satisfied, the restricted Delaunay triangulation becomes homeomorphic to the sampled surface. The result of Amenta and Bern says that if the sample is dense enough, the TBP is satisfied. Therefore, if one samples the surface Σ with locally furthest points until all triangles have small circumradius (that is, sample is dense), one is guaranteed to terminate with a surface mesh that is homeomorphic to Σ.

Cheng et al. [5] applied the above strategy to a specific type of smooth surface called skin surface. Boissonnat and Oudot [3] showed how the above strategy can be applied to the more general smooth surfaces. In doing so, they also put forward several results for surface sampling. The only drawback of their algorithm is that the user needs to compute the local feature sizes at sampled points. These are not easily computable in general. Cheng, Dey, Ramos, and Ray [6] proposed a different strategy that directly seeks for TBP violations to sample new points. This, however, needs to compute critical points of certain functions on the surface, which may not be easily computable. However, in a recent book on Delaunay mesh generation, Cheng, Dey, and Shewchuk have suggested a strategy that is more practical which leverages on both algorithms of Boissonnat and Oudot [3], and Cheng et al. [6]. We have the following result with this new algorithm:

Theorem 5. *There is a Delaunay refinement algorithm that runs with a parameter $\lambda > 0$ on an input smooth, compact, boundary-less surface Σ with the following guarantees:*

(1) *The output mesh is a Delaunay subcomplex and is a 2-manifold for all values of λ.*
(2) *If λ is sufficiently small, then the output mesh has similar guarantees with respect to the input surface Σ as in Theorem 4.*

A more precise statement of the above theorem is available from Chapter 14 of the book [7]. Oudot, Rineau, and Yvinec [10] extended the algorithm of Boissonnat and Oudot to meshing of volumes bounded by smooth surfaces. An improved version of this algorithm with new analysis is also available from [7]. The more difficult input such as piecewise smooth surfaces, and even piecewise smooth complexes have also been considered, and provable algorithms exist for them [4]. It is not possible to cover the substantial literature that have been developed for Delaunay mesh generation in this short abstract. We recommend the readers to the recent book on this topic [7].

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Relative Torsion in Homology Computations

Anil N. Hirani

(joint work with Tamal K. Dey, Bala Krishnamoorthy)

We describe a polynomial time algorithm for finding a smallest integer-valued chain homologous to a given chain. The size of the chain is measured as $\sum_i w_i x_i$ where $w_i \geq 0$ is a real number which is the weight of simplex number i and x_i is the value of the chain x on simplex i. If $w_i > 0$ then $\sum_i w_i x_i$ can be written more succinctly as $\|x\|_1$, the 1-norm of the chain x.

Some computations in topology start with first simplifying the given complex or the topological space. This is done in a way that preserves the invariants or classification outcome that is eventually sought. This methodology of simplifying first is very useful for algorithms, such as normal surface based algorithms. Without the simplification some algorithms would be prohibitively expensive for large complexes. Simplification has also proved to be a good heuristic even for problems where eventually a polynomial time algorithm is applied after simplification. An example of this class is Betti number computation.

In contrast, for the problem addressed here an answer is sought in a *given* simplicial complex. It is an optimization problem with a topological constraint. The problem statement is as follows:

> Given a simplicial complex K and a chain $c \in C_p(K; \mathbb{Z})$, find a p-chain x homologous to c such that x has the smallest total weight.

We call this the optimal homologous chain problem (OHCP). What makes this problem particularly interesting is that if $\mathbb{Z}$ is replaced by $\mathbb{Z}_2$ the above problem was shown to be NP-hard [1]. In fact, Chen and Freedman showed that it is NP-hard even to find an approximate solution to within any given constant factor. In [2] we showed that when integer homology is used, a polynomial time algorithm can be given for OHCP for a class of complexes. For example all orientable manifolds and d-dimensional complexes embedded in $\mathbb{R}^d$ are included in this class.

We found this class of complexes by finding a precise characterization of complexes for which the boundary matrix is totally unimodular (TU). (An integer matrix with entries in $\{0, 1, -1\}$ is TU if every square submatrix has determinant 0,1, or -1.) From the early days of integer linear programming the importance of the TU property has been well-known. When the constraint matrix in a linear program is TU, the polytope is integral for any integral right hand side of the constraint. That is, if a matrix A is TU and b an integral vector, the polytope $\{x \in \mathbb{R}^n \mid Ax \geq b\}$ is integral. In particular, this means that its vertices have integer coordinates. Thus the linear programming relaxation of the harder integer linear program results in an integer solution. Since linear programming has a polynomial time algorithm, this means that any linear program with a TU constraint matrix and integer b be solved in polynomial time while yielding an integer solution. This is relevant to OHCP because the constraint of x being homologous to c can be written as $x = c + \partial_{p+1} y$. Thus the boundary matrix ∂_{p+1} being TU is important here.

Our main theorem in [2] is that a boundary matrix is totally unimodular if and only if the complex is relatively torsion free. The precise statement is as follows:

> *For a finite simplicial complex K of dimension greater than p, the boundary matrix ∂_{p+1} is totally unimodular if and only if $H_p(L, L_0)$ is torsion-free, for all pure subcomplexes L_0, L in K of dimensions p and $p+1$ respectively, where $L_0 \subset L$.*

Intuitively, relative homology is relevant because when testing for total unimodularity, one is selecting submatrices. For the proof and more details see [2]. Besides solving the optimal homologous chain problem our result has some other unexpected consequences. For example, suppose one wants to check the presence of relative torsion in $H_p(L, L_0)$ for *all* pure subcomplexes $L_0 \subset L$ of dimension p and $p+1$. The naive algorithm for this is clearly exponential time since one has to select all possible subcomplexes. The test for whether $H_p(L, L_0)$ has torsion for a *given* pair is polynomial time. However the test over all subcomplexes would ostensibly require exponential time. But our result along with [3] implies that the test for presence or absence of relative torsion in $H_p(L, L_0)$ over all pure subcomplexes can be done in polynomial time. This is because [3] showed that there is a polynomial time algorithm to test whether a given matrix is TU or not.

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Solving Thurston's Equation Over a Commutative Ring

Feng Luo

Given a triangulated oriented 3-manifold or pseudo-3-manifold $(M, \mathcal{T})$, Thurston's equation associated to $\mathcal{T}$ is a system of integer coefficient polynomial equations defined on the triangulation. These polynomial equations are derived from the basic properties of the cross ratio. William Thurston introduced his equation in the field $\mathbf{C}$ of complex numbers in order to find hyperbolic structures. Since then, there has been much research on solving Thurston equation over $\mathbf{C}$ in the work of Choi, Neumann–Zagier, Petronio–Weeeks, Tillmann, Yoshida and others. Since the equations are integer coefficient polynomials, one could attempt to solve Thurston equation in a ring with identity. The purpose of this talk (based on the preprint [1]) is to show that interesting topological results about the 3-manifolds can be obtained by solving Thurston equation in a commutative ring with identity. For instance, one sees easily that Thurston equation is solvable in the field $\mathbf{Z}/3\mathbf{Z}$ of three elements if and only if each edge of the triangulation has even degree.

Main theorem. *Suppose $(M, \mathcal{T})$ is an oriented connected closed 3-manifold with a triangulation $\mathcal{T}$ and R is a commutative ring with identity. If Thurston equation on $(M, \mathcal{T})$ is solvable in R, then there exists a homomorphism ρ from $\pi_1(M)$ to $PGL(2, R)$ so that $\rho([e]) \neq id$ for each edge $e \in \mathcal{T}$ which is a loop. In particular, if $\mathcal{T}$ contains at most three vertices, then M is not simply connected.*

We remark that the existence of an edge which is a loop cannot be dropped in the theorem. Indeed, it is easy to see that for simplicial triangulations $\mathcal{T}$, there are always solutions to Thurston equation over $\mathbf{C}$. The main theorem for $R = \mathbf{C}$ was first proved by Segerman–Tillmann in [2]. A careful examination of the proof of [2] shows that their method also works for any field R. However, for a commutative ring with zero divisors, the geometric argument breaks down. We prove the main theorem by introducing a homogeneous Thurston equation and studying its solutions. The main theorem prompts us to introduce the universal construction of a Thurston ring of a triangulated 3-manifold. One can rephrase the main theorem in terms of the Thurston ring.

As a consequence of the main theorem, one obtains a result of Rubinstein and Tillmann that a closed 1-vertex triangulated 3-manifold is not simply connected if each edge has even degree. Solving Thurston's equation over a finite commutative ring produces many interesting combinatorial properties of the triangulation. For instance, Thurston's equation is solvable over the field of four elements if and only if one can color each tetrahedron red or black so that for each edge e, the number of red tetrahedra adjacent to e plus twice the number of black tetrahedra adjacent to e is divisible by 3.

Motivated by the main theorem, we propose the following two conjectures.

Conjecture 1. If M is a compact 3-manifold and $\gamma \in \pi_1(M) - \{1\}$, there exists a finite commutative ring R with identity and a homomorphism from $\pi_1(M)$ to $PSL(2, R)$ sending γ to a non-identity element.

A weaker form of the conjecture is,

Conjecture 2. If $M \neq S^3$ is a closed oriented 3-manifold, then there exists a 1-vertex triangulation $\mathcal{T}$ of M and a finite commutative ring R with identity so that Thurston equation associated to $\mathcal{T}$ is solvable in R.

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A Combinatorial Version of Homotopy for Simplicial Complexes

Jonathan A. Barmak
(joint work with Elias Gabriel Minian)

It is well known that we can associate a simplicial complex with any finite poset X. The simplices of the order complex $\mathcal{K}(X)$ are the non-empty chains of X. Conversely, given a finite simplicial complex K, there is a poset $\mathcal{X}(K)$, the face poset of K, which is the poset of simplices of K ordered by inclusion. A poset X can be thought of as a topological space whose points are the elements of X and whose open sets are the down-sets (sets closed by taking smaller elements). In fact such finite spaces are T_0 meaning that any two points can be separated with an open set. Moreover, any finite T_0 space is a poset in the way described above. A map between two posets is continuous if and only if it is order preserving. There is also a nice characterization of homotopies in this context due to Stong [9]. Two maps $f, g : X \to Y$ between finite spaces are homotopic if and only if there exists a sequence $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$ of maps from X to Y. Here, $f_0 \leq f_1$ means that the inequality holds pointwise.

The connection between posets and complexes is not just combinatorial but topological. A result of McCord [6] shows that there are weak homotopy equivalences $\mathcal{K}(X) \to X$ and $K \to \mathcal{X}(K)$, and, in particular a finite space and its order complex have the same homotopy and homology groups and the same is true for a complex and its face poset. If two finite spaces are homotopy equivalent, their order complexes are weak homotopy equivalent and then, being CW-complexes, they are homotopy equivalent. However, Whitehead's Theorem fails for finite spaces and there are examples of homotopy equivalent complexes with non homotopy equivalent face posets. In this talk I will introduce a new notion of homotopy for simplicial complexes and some results of a joint work with Gabriel Minian [1]. The *strong homotopy types* are the analogous of homotopy types but for the more restrictive notion of homotopy given by contiguity classes of simplicial maps. Strong homotopy types correspond exactly to homotopy types of finite spaces.

Theorem. If X and Y are homotopy equivalent finite spaces, then $\mathcal{K}(X)$ and $\mathcal{K}(L)$ are strong homotopy equivalent. On the other hand, if two complexes have the same strong homotopy type, their face posets are homotopy equivalent finite spaces.

Alternatively, strong homotopy types can be described, similarly to Whitehead's simple homotopy types, through a notion of collapse. A *strong collapse* is a sequence of deletions of vertices whose links are simplicial cones. Strong collapses preserve the strong homotopy type. A subcomplex L of a complex K is called a *core* of K if K strong collapses into L and the latter cannot be strong collapsed into a proper subcomplex. Using ideas similar to Stong's description of homotopy types of finite spaces, we proved the following:

Theorem. The core of a complex is unique up to isomorphism and two complexes have the same strong homotopy type if and only if their cores are isomorphic.

This result highlights an essential difference between strong and usual homotopy types. The contractibility of a finite simplicial complex is a problem which is algorithmically undecidable. However, there does exist an algorithm for deciding whether two complexes have the same strong homotopy type.

It is easy to see that strong collapses are particular cases of simplicial collapses and of non-evasive reductions, and as in those cases, strong collapsibility depends on the triangulation. The boundary of the 3-dimensional cross polytope minus a facet is a non strong collapsible triangulation of a 2-simplex. However, a big advantage of this new notion is the uniqueness of cores. There are examples of collapsible complexes, in the usual sense, which collapse to subcomplexes that are not collapsible. In contrast, if a strong collapsible complex strong collapses into a subcomplex, this one will also be strong collapsible. The uniqueness of cores is proved also in a purely combinatorial note by Matoušek [5].

We mention also connections between strong homotopy types and two important open conjectures. Recall that a complex K is said to be vertex homogeneous if for any two vertices $v, w \in K$ there exists a simplicial automorphism of K that maps v to w. A simplex is an example of a vertex homogeneous complex which is contractible. There exist contractible vertex homogeneous complexes which are not simplices [2, 3, 4]. The Evasiveness conjecture for simplicial complexes states that if we replace contractibility by non-evasiveness, then, there are no examples other than simplices. We proved that if we strengthen the hypothesis a little more, the statement is true.

Theorem. If a complex is strong collapsible and vertex homogeneous, then it is a simplex.

The second open problem we consider is Quillen's conjecture on the poset of p-subgroups of a group. If G is a finite group, $S_p(G)$ denotes the poset of non-trivial p-subgroups of G ordered by containment. If G has a non-trivial normal p-subgroup, then the order complex $\mathcal{K}(S_p(G))$ is contractible. Quillen conjectures in [8] that the converse is true. Using the notion of strong collapsible complex, we proved that the following is a restatement of Quillen's conjecture.

Restatement of Quillen's conjecture. If $\mathcal{K}(S_p(G))$ is contractible, then it is strong collapsible.

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Towards Algebraic Theory of Polytopes

Joseph Gubeladze

This research focuses on the category of convex polytopes and affine maps. While not exactly brand new, the line of research has potential to shed new light on some of the central concepts in polytope theory and propose natural extensions, reminiscent to the triangulations vs. homology dichotomy in topology.

Understanding *hom-polytopes* (a.k.a. *mapping polytopes*) – the polytope of all affine maps between two given polytopes – is the first step in the proposed categorial analysis. The software polymake has a special module for computing these polytopes. But explicit computations become increasingly more difficult as the combinatorics of polytopes in question gets richer. The challenge here is to understand the vertices of hom-polytopes. But, currently, even the case of general regular source and target polygons seems to be out of reach. The first systematic analysis of the vertex sets of hom-polytopes in various settings is done in [2]. This includes polytopal analogs of the rank-nullity theorem and generic or regular source/target polygons.

One quickly runs into paradigmatic challenges when the next natural step in the direction outlined above is attempted. Below are details.

The Billera–Sturmfels notion of fiber polytopes [1], which plays an important role in the theory of triangulations, is expected to be the right kernel object in the category of polytopes. But it is not quite clear what the appropriate formal framework should be. One possibility (proposed by Eric Katz) is to prove that there is a natural isomorphism

$$\mathrm{Hom}(P, \Sigma f) = \Sigma\,\mathrm{Hom}(P, f), \tag{1}$$

where P is a polytope and f is an affine map between two polytopes. (One needs to formally extend the fiber construction to not necessarily surjective maps.) This would mimic the universal equality in a general category

$$\mathrm{Hom}(a, \varprojlim \mathcal{D}) = \varprojlim \mathrm{Hom}(a, \mathcal{D}), \tag{2}$$

a being an object and $\mathcal{D}$ a diagram. While the expected isomorphism (1) is interested in its own right, more important is that it suggests an approach to *quotient polytopes* – the conjectural dual objects to the fiber polytopes. In fact, (2) dualizes as

$$\mathrm{Hom}(\varinjlim \mathcal{D}, a) = \varprojlim \mathrm{Hom}(\mathcal{D}, a), \tag{3}$$

and the polytopal version of (3) should look like

$$\mathrm{Hom}(\mathrm{Coker} f, P) = \Sigma\,\mathrm{Hom}(f, P), \tag{4}$$

where $f : Q \to R$ is an affine map and $\mathrm{Coker} f$ is the corresponding enigmatic cokernel/quotient object. To put in plain polytopal terms, even though we do not know how to construct quotient polytopes, categorial analysis suggests what the hom-polytope – an actual polytope – from the quotient to *any* polytope P should be. Now, this almost certainly puts the quotient polytopes outside of the category of genuine polytopes because the contravariant functor $\mathrm{Hom}(\mathrm{Coker} f, -)$ is unlikely to be representable. But the 'abstract nonsense' machinery provides with a recipe for approximating (as a universal limit) $\mathrm{Coker} f$ by actual polytopes. A way to keep track of concrete polytopal consequences this constructions can have is to relate it to triangulations, much like the fiber construction relates to regular triangulations.

The universal approach we are advocating here consists of the two steps:

- Construct a functor from polytopes to sets with properties reminiscent of those the functor, represented by the conjectural object, should have;
- Explicate this functor as limit of functors, representable by genuine polytopes.

Such an approach is rather robust: should (1) fail, there is a dual setup for introducing Coker$f$, independent of a prerequisite equality. The idea is to introduce hom-sets *from actual polytopes to the quotient polytopes.* These sets are real algebraic varieties in a natural way, and they have been studied already in the totally different context of conditional independence in statistics [3].

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Metric Geometry and Random Discrete Morse Theory

Bruno Benedetti
(joint work with Karim Adiprasito, Frank Lutz)

The exposition is in two parts. In the first part, I sketch the main result of the recent preprint [1] on metric geometry and collapsibility (with Karim Adiprasito). In the second part, I present a possible computational approach (with Frank Lutz).

1. Metric geometry on simplicial complexes

How to put a metric on a given simplicial complex? One way is to declare all edges to have unit length, and to regard all triangles as equilateral triangles in the Euclidean plane. This yields the *equilateral flat* metric, also known as *regular* metric. Many other options are possible; for example, one can assign different lengths to the various edges. The metric is called *acute* (resp. *non-obtuse*) if all dihedral angles in each simplex are less than 90 degrees (resp. at most 90 degrees). Clearly, equilateral implies acute, which in turn implies non-obtuse.

CAT(0) spaces. Once endowed with such metric, a simplicial complex becomes a *geodesic space*, i.e. a locally compact metric space in which distances can be measured along shortest paths. A geodesic space is called CAT(0) if any triangle formed by three shortest paths looks not fatter than the "corresponding" triangle (=with same edge lengths) in $\mathbb{R}^2$. For example, the cylinder $S^1 \times [0, 1]$ is *not* CAT(0): Any non-degenerate triangle formed by three points on $S^1 \times \{0\}$ and by the shortest paths connecting them, is fatter than the corresponding Euclidean triangle. (Spaces where every point has a CAT(0) neighborhood, like the cylinder, are called *non-positively curved.*) All CAT(0) spaces are contractible. Apart from dimension one, the converse is false, as shown by the Dunce Hat.

Collapsible complexes. A well known combinatorial strengthening of contractibility was introduced in 1939 by Whitehead. A *free face* in a complex is a face properly contained only in one other face. (Not all complexes have free faces.) A complex is called *collapsible* if it can be reduced to a point by recursively deleting a free face. The deletion of a free face (and of the other face containing it) is topologically a deformation retract. Hence, all collapsible complexes are contractible. Apart from dimension one, the converse is false, as shown by the Dunce Hat.

It is easy to see that the stellar subdivision of a triangle is collapsible. However, it is not CAT(0) with the equilateral flat metric, basically because of the central degree-3 vertex. In 2008, Crowley found a first non-trivial relation between the aforementioned properties:

Theorem 1 (Crowley). *Every* 3*-dimensional simplicial pseudomanifold that is* CAT(0) *with the equilateral flat metric, is collapsible.*

It turns out that a more general fact is true:

Main Result 1 (Adiprasito–B. [1]). *Every d-dimensional polytopal complex that is* CAT(0) *with a metric for which all vertex stars are convex, is collapsible.*

Under the equilateral flat metric, it is easy to see that all vertex stars are convex. Actually, even in any *non-obtuse* flat metric, all vertex stars are convex. This simple observation has three main consequences. Recall that CAT(0) *cube complexes* are CAT(0) spaces obtained from cubical complexes by giving each cube the metric of a regular Euclidean cube.

Consequence 1. *All* CAT(0) *cube complexes are collapsible.*

Consequence 2. *There is a smooth* 5*-manifold (with boundary) M so that*
- (i) *M is not homeomorphic to the d-ball.*
  Hence, every PL triangulation of M is not collapsible.
- (ii) *M is homeomorphic to a (compact)* CAT(0) *cube complex.*
  Hence, some non-PL triangulation of M is collapsible.

Consequence 3. *Discrete Morse inequalities may be sharper than (smooth) Morse inequalities, in bounding the homology of a manifold. Also, non-PL structures may be more efficient than PL structures, from a computational point of view.*

2. A computational approach: Random discrete Morse theory.

After proving that not all collapsible manifolds are balls, a natural question is whether it is possible to construct one example explicitly. This raises complexity issues. In principle, collapsibility is algorithmically decidable; one could just try all possible sequences of free-face-deletions. However, already for 3-balls with, say, 20 facets, the number of all possible sequences is beyond the computational limit.

A related problem is how to tell whether a triangulation is 'nice', and how to quantify its nastyness. If we decided to regard collapsible (or shellable) balls as the nicest triangulations, what is 'far from being nice'?

A possible statistic approach is what we sketch below, called *random discrete Morse theory* [3]. The idea is elementary, and consists of the following algorithm:

Input: An arbitrary simplicial complex C, given by it list of facets (or by its face poset). Initialize $c_0 := 0$, $c_1 := 0$, $\dots$, $c_{\dim C} := 0$.
(1) Is the complex empty? If yes, stop the algorithm; otherwise, go to (2).
(2) Are there free codimension-one faces? If yes, go to (3); if no, go to (4).
(3) (Elementary Collapse): Pick one free codimension-one face *uniformly at random* and delete it. Then go back to (1).
(4) (Storage of Critical Face): Pick one of the top-dimensional faces *uniformly at random* and delete it from the complex. If i is the dimension of the deleted face, increment c_i by 1 unit; then go back to (1).
Output: the "discrete Morse vector" $(c_0, c_1, c_2, \dots, c_{\dim C})$. By construction, c_i counts critical faces of dimension i.

This algorithm requires no backtracking, and 'digests' the complex very rapidly. The output $(1, 0, 0, \dots, 0)$ is a certificate of collapsibility. If the output is different, the complex could still be collapsible with a different sequence of free-face deletions. Every sequence has some positive probability to be the one picked up by the algorithm; unfortunately, this probability can be arbitrarily small. Nevertheless, when the certificate of collapsibility is not reached, the algorithm outputs something meaningful, namely, the f-vector of a homotopy equivalent cell complex. Intuitively, if this vector is close to $(1, 0, 0, \dots, 0)$, we could still say that the complex is 'close to be collapsible'.

Since the output arrives quickly, we can re-launch the program, say, 10000 times, possibly on separate computers (independently). The distribution of the obtained outcomes yields an approximation of the *discrete Morse spectrum*, which is the distribution of all possible outcomes. This allows an empirical analysis of how complicated the complex is. For example, here is the data collected by running the algorithm 10000 times on Hachimori's triangulation `nc-sphere` [6]:

$\mathbb{Z}$-homology $= (\mathbb{Z}, 0, 0, \mathbb{Z})$, $\pi_1 = (0)$	$(1,1,1,1)$:	7902
f-vector $= (381, 2309, 3856, 1928)$	$(1,2,2,1)$:	1809
	$(1,3,3,1)$:	234
Time employed:	$(1,4,4,1)$:	25
3.228 seconds (Hasse diagram)	$\mathbf{(1,0,0,1)}$:	**12**
+0.470 seconds per run	$(2,3,2,1)$:	9
	$(1,6,6,1)$:	3
	$(2,4,3,1)$:	3
	$(2,5,4,1)$:	2
	$(1,5,5,1)$:	1

The optimal Morse vector appears in 0.12% of cases, so `nc-sphere` minus a facet is somewhat 'barely collapsible'. In fact, our non-deterministic algorithm is the first to find a collapsing sequence for it. In contrast, on many polytopal 3-spheres (and also for Barnette's non-polytopal, shellable 3-sphere) the Morse vector (1, 0, 0, 1) appears basically 100% of the times.

This way we obtained optimal discrete Morse functions for many triangulations of various topologies and dimensions, among which Kühnel et al.'s triangulations of the K_3 surface [4, 8] and of $\mathbb{CP}^2$, or the Csorba–Lutz triangulation of the Hom-complex $\mathrm{Hom}(C_5, K_5)$ [5]. The spectra which one can intuit experimentally seem interesting per se; we hope future theoretical work can justify them. Here is the largest example on which the algorithm was successful:

Main Result 2 (Adiprasito–B.–Lutz [2]). *There is a collapsible 5-manifold different from the 5-ball with f-vector* $(5013, 72300, 290944, 495912, 383136, 110880)$.

The construction is as follows. Start with the 16-vertex triangulation of the Poincaré sphere; remove the star of a vertex; take the product with an interval; cone over the boundary; form a one-point suspension to achieve a non-PL 5-sphere with 32 vertices; take the barycentric subdivision; take the collar of the PL singular set. The resulting manifold is homeomorphic to the one in [1, Thm. 4.12].

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Barycentric Subdivisions, Shellability and Collapsibility

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(joint work with Bruno Benedetti)

We report on recent results from the papers [1, 2].

1. Shellability and collapsibility of convex sets, applications

Shellability is one of the earliest notions in combinatorial topology. It is deeply connected to the theory of convex polytopes, via Bruggesser–Mani's result that the boundary of any convex polytope, as a polytopal complex, is shellable. Collapsibility is a notion due to Whitehead, and lies at the core of simple homotopy theory. In this talk, we investigate a question going back to Lickorish:

Problem 1 (Lickorish cf. [6, 13]). *Let C be a linear triangulation of a convex ball in Euclidean space. Is it true that C is collapsible?*

This was verified up to dimension 3 by Chillingworth [6]. A linear triangulation will from now on be called a *subdivision*. A motivation for this question is the problem of whether collapsibility is preserved under subdivisions.

Problem 2 (Variant of Problem 1, cf. [11]). *Assume that C is some simplicial complex that collapses to a subcomplex C'. If D is some subdivision of C such that D restricts to a subdivision D' of C', is it true that D collapses to D'?*

A positive answer to Problem 1 does not immediately imply a positive answer to Problem 2. Problem 2 is equivalent to requiring the stronger conclusion of *endo-collapsibility* (that means boundary critical collapse, cf. [3]) in Problem 1.

The naive approach of shelling along a linear functional, does not work: In fact, there are linear triangulations of convex sets that are not shellable, as famously displayed by Rudin in 1957 [15]. Rudin's ball is a subdivision of a tetrahedron that is not shellable. We provide the following reconciliation:

Theorem 1 (A.–Benedetti 2012 [1, 2]). *If C is a subdivision of a convex d-ball,*
– *the barycentric subdivision of C is constructible, and in particular endocollapsible and collapsible, and*
– *the $(d-2)$-fold barycentric subdivision of C is shellable.*

In particular, any linearly triangulated convex 3-ball (e.g. Rudin's Ball) becomes shellable after a single barycentric subdivision. As a corollary, we obtain that collapsibility is almost preserved under subdivisions:

Corollary 2 (A.–B. [2]). *Let C be a simplicial complex that collapses to a subcomplex C'. Let D be a subdivision of C that restricts to a subdivision D' of C'. The barycentric subdivision of D collapses to the barycentric subdivision of D'.*

Corollary 3 (A.–B. [2]). *Let C be a PL d-ball. Then some r-fold barycentric subdivision of C is shellable, and in particular collapsible.*

This allows us to give a simple proof of the fact [4, 9] that discrete Morse Theory is as perfect as smooth Morse Theory. More precisely, if M is a closed smooth manifold and f a Morse function on M with c_i critical points of index i, then there exists a discrete Morse function on some PL triangulation of M that has at most c_i critical faces of dimension i. Combined with a fact in [1], this gives a surprising result: Discrete Morse Theory is not only at least as perfect as smooth Morse Theory, but in many cases even better: There exist, for example, contractible smooth manifolds which allow only a nontrivial handle decomposition, but that, at the same time, allow a triangulation that is collapsible.

Finally, we apply our methods to an old result by Dierker, Lickorish and Cohen:

Theorem 2 (Dierker [8], Lickorish [14], Cohen [7]). *If C is any contractible simplicial d-complex,* some *subdivision of $C \times I^{\max(5,2d)}$ is collapsible.*

We investigate whether one can strengthen this theorem by concluding that $C \times I^{\max(5,2d)}$, as polytopal complex, is (polytopally) collapsible. First, we use knot theory to provide a negative answer

Proposition 1 (A.–B. [2]). *For every nonnegative integer n, there exists a contractible simplicial 2-complex C such that $C \times I^n$ is not collapsible.*

Still, using the methods of Theorem 1, we can prove:

Theorem 3 (A.–B. [2]). *Let C be a contractible simplicial complex. Then, for some n, the polytopal complex $C \times I^n$ becomes collapsible.*

This confirms a conjecture of Bob Oliver. In particular, we obtain a simple construction method for counterexamples to a conjecture of Kahn, Saks and Sturtevant, which is originally due to Oliver:

Corollary 4 (Oliver '84, A.–B. [2]). *Some collapsible simplicial complexes, different than the simplex, have symmetry group that act transitively on their vertices.*

2. CAT(1) metrics and the Hirsch conjecture

The shellability property is related to diameter condition, as in the Hirsch bound. The connection goes back to a famous result of Provan and Billera [5]:

Theorem 4 (Provan & Billera '79). *Let C be any simplicial d-complex on n vertices ($d \geq 1$).*
- *If C is vertex-decomposable, then C satisfies the Hirsch conjecture, i.e. the diameter of its dual graph is bounded above by $n - d - 1$.*
- *if C is shellable, then the barycentric subdivision of C is vertex-decomposable.*

In particular, the barycentric subdivision of a connected shellable simplicial complex satisfies the diameter bound conjectured by Hirsch for polytopes. Using methods from metric geometry, we generalize this result to all simplicial complexes satisfying a natural connectivity assumption that is much weaker than shellability.

Recall that a length space is called CAT(1) if, roughly speaking, any triangle formed by three shortest paths is "not fatter" than the corresponding spherical triangle in the unit sphere. We wish to study simplicial complexes which allow a CAT(1)-metric.

Theorem 5 (Gromov [10, Theorem 4.2.A]). *For a simplicial complex C, the following are equivalent:*
(i) *C is* CAT(1) *with the all-right metric;*
(ii) *C is* flag*, that is, its minimal non-faces are edges.*

Here we present an application of Gromov's theorem to the Polynomial Hirsch Conjecture. Recall that a complex is *superstrongly connected* if the link of every face (of codimension at least 2) in it is strongly connected.

Conjecture 2 (Polynomial Hirsch Conjecture). *Consider the family $\mathfrak{F}_{n,d}$ of superstrongly-connected d-dimensional simplicial complexes with n vertices. There is a polynomial P in two variables such that the diameter of the dual graph of any complex in $\mathfrak{F}_{n,d}$ is at most $P(n, d)$.*

What we achieve is a proof of this conjecture for flag complexes:

Theorem 6 (A.–B. [1]). *Let C be a superstrongly connected d-complex with n vertices.*

(i) *If C is flag, then the diameter of the dual graph of C is at most $n-d-1$.*

(ii) *If all minimal non-faces of C have dimension $\leq i$, then the diameter of the dual graph of C is at most $i!\binom{n}{i-1}+n-d-1$.*

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Centrally Symmetric Polytopes with Many Faces

Isabella Novik

(joint work with Alexander Barvinok, Seung Jin Lee)

A *polytope* is the convex hull of a set of finitely many points in R^d. A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if $P=-P$. The question we are interested in is "What is the maximum number of k-dimensional faces that a centrally symmetric d-dimensional polytope with N vertices can have?" We denote this number by $\operatorname{fmax}_k(d,N)$.

Our interest in this question stems from a variety of reasons. One of them is that while the answer in the class of all polytopes is classic by now [7], very little is known in the centrally symmetric case. Another one comes from the desire to find new minimal triangulations of manifolds: many known minimal triangulations of manifolds (see [8], [6]) were constructed as subcomplexes of the boundary complex of the cyclic polytope; this makes one hope that cs polytopes with many faces will give rise to new constructions of centrally symmetric triangulations of manifolds. (Here by a cs triangulation of a manifold M we mean a simplicial complex with a fixed point free involution whose geometric realization is M). Yet another motivation comes from work of Donoho and his collaborators (see [4]) who discovered that centrally symmetric polytopes with many faces have implications in problems of sparse signal reconstruction.

While we are still very far from understanding the exact value of $\mathrm{fmax}_k(d,N)$, we have established the following bounds on it:

Theorem 1. *The maximum possible number of edges in a cs polytope satisfies*

$$\left(1-\frac{4}{(\sqrt{3})^d}\right)\binom{N}{2} \le \mathrm{fmax}_1(d,N) \le \left(1-\frac{1}{2^d}\right)\frac{N^2}{2},$$

and the maximum number of $(k-1)$-dimensional faces satisfies

$$\left(1-\frac{k^2}{(\gamma_k)^d}\right)\binom{N}{k} \le \mathrm{fmax}_{k-1}(d,N) \le \left(1-\frac{1}{2^d}\right)\binom{N}{k}\cdot\frac{N}{N-1},$$

where $\gamma_k = 2^{3/20k^2 2^k}$ for $3 \le k \le d/2$.

The upper bounds in the above theorem were obtained in [1] using the volume trick. The lower bounds are very recent [3] and are obtained via explicit construction. In the rest of this presentation we provide a sketch of this construction.

Recall that the $2k$-dimensional *cyclic polytope* with N vertices, $C_{2k}(N)$, is defined as the convex hull of N distinct points on the trigonometric curve

$$M_k : \mathbb{R} \longrightarrow \mathbb{R}^{2k}, \quad M_k(t) = (\cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \ldots,\ \cos kt,\ \sin kt)\,.$$

Since

$$M_k(t) = M_k(t+2\pi) \quad \text{for all} \quad t,$$

from this point on, we consider $M_k(t)$ to be defined on the unit circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. The celebrated Upper Bound Theorem [7] asserts that among all $2k$-dimensional polytopes on N vertices, the cyclic polytope simultaneously maximizes all the face numbers.

In an attempt to construct a centrally symmetric analog of the cyclic polytope, consider the *symmetric moment curve* $U_k : \mathbb{S} \longrightarrow \mathbb{R}^{2k}$ defined by

$$U_k(t) = (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t,\ \ldots,\ \cos(2k-1)t,\ \sin(2k-1)t)\,.$$

Observe that $U_k(t+\pi) = -U_k(t)$, and so the convex hull of U_k, $\mathcal{B}_k := \mathrm{conv}(U_k(t) : t \in \mathbb{S})$, is a cs convex body. The curve U_2 and its convex hull $\mathcal{B}_2$ were originally considered by Smilansky [9].

For an integer $m \ge 1$, consider the curve

$$\Phi_m : \mathbb{S} \longrightarrow \mathbb{R}^{2(m+1)}, \quad \Phi_m(t) := \big(\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(3^m t), \sin(3^m t)\big).$$

Note that $\Phi_1 = U_2$.

The lower bound on the maximum possible number of edges in a cs polytope of Theorem 1 is a consequence of the following construction/result. We say that a cs polytope is 2-*neighborly* if every two of its vertices that are not antipodes form the vertex set of an edge.

Theorem 2. *Fix integers $m \geq 2$ and $s \geq 2$. Let $A_m \subset \mathbb{S}$ be the set of $2(3^m - 1)$ equally spaced points:*

$$A_m = \left\{ \frac{\pi(j-1)}{3^m - 1} : \quad j = 1, \dots, 2(3^m - 1) \right\},$$

and let $A_{m,s} \subset \mathbb{S}$ be the set of $2(3^m - 1)$ clusters of s points each, chosen in such a way that for all $j = 1, \dots, 2(3^m - 1)$, the j-th cluster lies on an arc of length 10^{-m} that contains the point $\frac{\pi(j-1)}{3^m-1}$, and the entire set $A_{m,s}$ is centrally symmetric. Then

(1) *The polytope $\operatorname{conv}(\Phi_m(t) : t \in A_m)$ is a centrally symmetric 2-neighborly polytope of dimension $2(m+1)$ that has $2(3^m - 1)$ vertices.*
(2) *The polytope $\operatorname{conv}(\Phi_m(t) : t \in A_{m,s})$ is a centrally symmetric $2(m+1)$-dimensional polytope with $N := 2s(3^m - 1)$ vertices and such that every two of its vertices that do not lie in a pair of opposite clusters form the vertex set of an edge. In particular, this polytope has at least*

$$N(N - s - 1)/2 > \left(1 - 3^{-m}\right) \binom{N}{2}$$

edges.

We refer our readers to [3] for the proof of Theorem 2. Here we only mention that one of the main ingredients of the proof is a result of Smilansky [9] asserting that for every two points $t_1, t_2 \in \mathbb{S}$ that lie on an open arc of length $2\pi/3$, the line segment $[U_2(t_1), U_2(t_2)]$ is an (exposed) edge of $\mathcal{B}_2$.

Our construction of cs polytopes with many $(k-1)$-dimensional faces is more complicated, but utilizes similar ideas. The details can be found in [3]. Here we only note that this construction is based on the curve $\Psi_{k,m} : \mathbb{S} \longrightarrow \mathbb{R}^{2k(m+1)}$ defined by

$$\Psi_{k,m}(t) := \left(U_k(t), U_k(3t), U_k(3^2 t), \dots, U_k(3^m t)\right)$$

instead of Φ_m, and on the set of points $V(\mathcal{F}) \subset \mathbb{S}$ (clusters, resp.) associated with a large k-*independent* family $\mathcal{F}$ of subsets of $[m] := \{1, 2, \dots, m\}$ instead of A_m ($A_{m,s}$, resp.). A family $\mathcal{F}$ is called k-independent if for every k distinct subsets $I_1, \dots, I_k$ of $\mathcal{F}$ all 2^k intersections

$$\bigcap_{j=1}^{k} J_j, \quad \text{where } J_j = I_j \text{ or } J_j = I_j^c := [m] \setminus I_j, \text{ are non-empty.}$$

The crucial component of our construction is a deterministic construction of k-independent families of size larger than $2^{m/5(k-1)2^k}$ given in [5]. Another component is a result from [2] asserting that for every $s \leq k$ and every choice of s points $t_1, \dots, t_s \in \mathbb{S}$ that lie on a closed arc of length $\pi/2$, the set $\{U_k(t_1), \dots, U_k(t_s)\}$ is the vertex set of an exposed face of $\mathcal{B}_k$.

Remark 3. *The lower bounds of Theorem 1 become non-trivial only when* $k = O(\log d)$*. For higher values of* k *the following simple construction provides cs* d*-polytopes with* N *vertices and what appears to be the current record number of* $(k-1)$*-dimensional faces: let* $D \subset \mathbb{S}$ *be a cs set of four clusters of about* $N/4$ *points each with the* j*-th cluster lying on a small arc around* $\pi j/4$ *for* $j = 0, 1, 2, 3$*. Define* $P(d, N)$ *to be the convex hull of* $(U_{\lfloor d/2 \rfloor}(t) \; : t \in D)$ *if* d *is even and the bipyramid over* $P(d-1, N-2)$ *if* d *is odd. Then every* k *vertices of this polytope that belong to the union of two neighboring clusters form the vertex set of a* $(k-1)$*-face. In particular,* $P(d, N)$ *has at least*

$$4\binom{N/2}{k} - 4\binom{N/4}{k} \approx \left(\frac{1}{2^{k-2}} - \frac{1}{4^{k-1}}\right)\binom{N}{k}$$

$(k-1)$*-dimensional faces for all* $k < d/2$*.*

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Not all Simplicial Polytopes are Weakly Vertex-Decomposable

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(joint work with Jesús A. De Loera)

Due to its relevance to the theoretical performance of the simplex method of linear programming, a lot of effort has been invested in bounding the diameters of convex polyhedra. The 1957 Hirsch conjecture for polytopes became one of the most important problems in combinatorial geometry. For simple polytopes, the Hirsch conjecture stated that any two vertices in a simple d-polytope with n facets can be connected by an edge path of length at most $n-d$. We will work in the polar setting where the Hirsch conjecture for simplicial polytopes asserts that if P is a simplicial d-polytope with n facets, then any pair of facets in P can be connected by a facet-ridge path of length at most $n - d$. The Hirsch conjecture remained open until 2010 when Santos [10] constructed a 43-dimensional counterexample to the Hirsch conjecture with 86 vertices.

Despite this great success the best known counterexamples to the Hirsch conjecture have diameter $(1+\epsilon)(n-d)$, while the best known upper bounds on diameter are quasi-exponential in n and d [2] or linear in fixed dimension but exponential in d [1, 7]. Today we do not even know whether there exists a polynomial bound on the diameter of a polytope in terms of its dimension and number of vertices. Thus studying diameters of simplicial spheres and polytopes is still the subject of a great interest. In this report, which is a summary of the results in [4], we outline an approach of Provan and Billera [9] that can be used to give diameter bounds for so-called (weakly) k-decomposable simplicial complexes. After introducing these complexes, we will provide explicit examples of simplicial polytopes that are not even weakly vertex-decomposable which arise naturally as the polars of certain simple $2 \times n$ transportation polytopes.

1. Decomposability of simplicial complexes

One approach to trying to establish polynomial diameter bounds is to study decompositions of simplicial complexes. Provan and Billera [9] defined a notion of k-decomposability for simplicial complexes and showed that k-decomposable complexes satisfy nice diameter bounds.

Definition 1. ([9, Definition 2.1]) *A simplicial complex Δ of dimension $d-1$ is **k-decomposable** if Δ is pure and either*

(1) Δ is a simplex, or
(2) there is a face $\tau \in \Delta$ with $\dim \tau \leq k$ such that
  (a) $\Delta \setminus \tau$ is $(d-1)$-dimensional and k-decomposable and
  (b) $\mathrm{lk}_\Delta(\tau)$ is $(d-|\tau|-1)$-dimensional and k-decomposable.

Theorem 2. ([9, Theorem 2.10]) *Let Δ be a k-decomposable simplicial complex of dimension $d-1$. Then*

$$\mathrm{diam}(\Delta) \leq f_k(\Delta) - \binom{d}{k+1},$$

where $f_k(\Delta)$ denotes the number of k-dimensional faces in Δ.

In particular, a 0-decomposable complex (also called vertex-decomposable) satisfies the Hirsch bound. One approach to trying to prove the Hirsch conjecture would be to try to show that any simplicial polytope is vertex-decomposable. In his thesis, Lockeberg [8] constructed a simplicial 4-polytope on 12 vertices that is not vertex-decomposable (see also [6, Proposition 6.3]), though the diameter of Lockeberg's polytope still satisfies the Hirsch bound. Of course Santos' counterexample to the Hirsch conjecture also is not vertex-decomposable. In addition, Provan and Billera define a weaker notion of k-decomposability that does not require any condition on links but still provides bounds on the diameter of a simplicial complex.

Definition 3. *A simplicial complex Δ of dimension $d-1$ is **weakly k-decomposable** if Δ is pure and either*

(1) Δ is a simplex, or

(2) *there is a face $\tau \in \Delta$ with $\dim \tau \leq k$ such that $\Delta \setminus \tau$ is $(d-1)$-dimensional and weakly k-decomposable.*

Theorem 4. ([9, Theorem 4.2.3]) *Let Δ be a weakly k-decomposable simplicial complex of dimension $d-1$. Then*

$$\operatorname{diam}(\Delta) \leq 2 f_k(\Delta).$$

Again, we say that a weakly 0-decomposable complex is weakly vertex-decomposable. Based on the hope that diameters of simplicial polytopes have linear upper bounds, it would be natural to try to prove that any simplicial d-polytope is weakly vertex-decomposable (see Sections 5, 6, and 8 in [6] for additional discussion on decomposability and weak decomposability). In the next section, we will give a counterexample to this conjecture by introducing a family of simple transportation polytopes whose polars are not even weakly vertex-decomposable.

2. Transportation polytopes

For fixed vectors $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, the **classical $\mathbf{m} \times \mathbf{n}$ transportation polytope $\mathbf{P}(\mathbf{a}, \mathbf{b})$** is the collection of all nonnegative matrices $X = (x_{i,j})$ with $\sum_{i=1}^m x_{i,j} = b_j$ for all $1 \leq j \leq n$ and $\sum_{j=1}^n x_{i,j} = a_i$ for all $1 \leq i \leq m$. The vectors $\mathbf{a}, \mathbf{b}$ are often called the **margins** of the transportation problem.

There is a natural way to associate a complete bipartite graph $K_{m,n}$ with weighted edges to each matrix $X \in P(\mathbf{a}, \mathbf{b})$ by placing a weight of $x_{i,j}$ on the edge $(i,j) \in [m] \times [n]$. We summarize the properties of transportation polytopes that we will use in the following theorem. These results and their proofs can be found in [5].

Theorem 5. *Let $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$ with $mn > 4$.*

(1) *The set $P(\mathbf{a}, \mathbf{b})$ is nonempty if and only if $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$.*
(2) *The dimension of $P(\mathbf{a}, \mathbf{b})$ is $(m-1)(n-1)$.*
(3) *The transportation polytope $P(\mathbf{a}, \mathbf{b})$ is nondegenerate (hence simple) if and only if the only nonempty sets $S \subseteq [m]$ and $T \subseteq [n]$ for which $\sum_{i \in S} a_i = \sum_{j \in T} b_j$ are $S = [m]$ and $T = [n]$.*
(4) *If $P(\mathbf{a}, \mathbf{b})$ is simple, the set*

$$F_{p,q} = F_{p,q}(\mathbf{a}, \mathbf{b}) := \{X \in P(\mathbf{a}, \mathbf{b}) : x_{p,q} = 0\},$$

is a facet of $P(\mathbf{a}, \mathbf{b})$ if and only if $a_p + b_q < \sum_{i=1}^m a_i$.
(5) *If $P(\mathbf{a}, \mathbf{b})$ is simple, the matrix $X \in P(\mathbf{a}, \mathbf{b})$ is a vertex of $P(\mathbf{a}, \mathbf{b})$ if and only if the edges $\{(i,j) \in K_{m,n} : x_{i,j} > 0\}$ form a spanning tree of $K_{m,n}$.*

Our study focuses on a certain family of simple $2 \times n$ transportation polytopes whose polars are not weakly vertex-decomposable. One might hope that these polytopes might also provide natural counterexamples to the Hirsch conjecture since they are not weakly vertex-decomposable. This is not the case, however, since Kim [3, Theorem 3.5.1] showed that any classical $2 \times p$ transportation polytope satisfies the Hirsch conjecture.

Theorem 6. ([4, Theorems 3.1, 3.2]) *For all $d \geq 4$, there exists a simplicial d-polytope that is not weakly vertex-decomposable:*

(a). For all $n \geq 2$, the simplicial polytope $P^{\Delta}(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = (2n+1, 2n+1)$ and $\mathbf{b} = (2, 2, \ldots, 2) \in \mathbb{R}^{2n+1}$ is not weakly vertex-decomposable.
(b). For all $n \geq 3$, the simplicial polytope $P^{\Delta}(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = (2n-1, 2n+1)$ and $\mathbf{b} = (2, 2, \ldots, 2) \in \mathbb{R}^{2n}$ is not weakly vertex-decomposable.

We conclude with some questions related to decomposability of simplicial complexes that we feel are interesting both in their relation to the polynomial Hirsch conjecture and in their own right.

It is obvious that any k-decomposable simplicial complex is also $(k+1)$-decomposable; and Provan and Billera [9, Theorem 2.8] prove that a $(d-1)$-dimensional simplicial complex is $(d-1)$-decomposable if and only if it is shellable.

Question 7. *Any simplicial d-polytope is shellable, but there exist simplicial polytopes that are not (weakly) 0-decomposable. What can be said about the minimal value $k = k(d)$ such that all simplicial d-polytopes are (weakly) k-decomposable, but there exist simplicial d-polytopes that are not (weakly) $(k-1)$-decomposable?*

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Catalogues of PL-Manifolds and Complexity Estimations via Crystallization Theory

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1. Abstract

Crystallization theory is a graph-theoretical representation method for compact PL-manifolds of arbitrary dimension, with or without boundary, which makes use of a particular class of edge-coloured graphs, which are dual to coloured (pseudo-) triangulations. These graphs are usually called *gems*, i.e. *Graphs Encoding Manifolds*, or *crystallizations* if the associated triangulation has the minimal number of vertices.

One of the principal features of crystallization theory relies on the purely combinatorial nature of the representing objects, which makes them particularly suitable for computer manipulation.

The present talk focuses on up-to-date results about:

- generation of catalogues of PL-manifolds for increasing values of the vertex number of the representing graphs;
- definition and/or computation of invariants for PL-manifolds, directly from the representing graphs.

2. Cataloguing PL-manifolds via crystallization theory

Tables of crystallizations have been obtained in dimension 3, and are in progress in dimension 4: the main tool for their generation is the *code*, a numerical "string" which completely describes the combinatorial structure of a coloured graph, up to colour-isomorphisms [12]. Afterwards suitable moves on gems, translating the PL-homeomorphism of the represented manifolds, are applied to develop a classification procedure which allows to detect crystallizations of the same manifold; this is the starting point toward the identification of the manifolds represented in the catalogues (see [7] and related C++ programs – jointly elaborated with P. Cristofori – whose codes have been recently parallelized in order to obtain a better performance[1]).

It is worthwhile noting that in dimension 3 the above automatic partition into equivalence classes succeeds to distinguish topologically all manifolds represented by the generated catalogues. This allows to classify the 110 (resp. 16) closed prime orientable (resp. non-orientable) 3-manifolds having a coloured triangulation with at most 30 tetrahedra. The obtained results comprehend the JSJ-decomposition of all involved manifolds, together with the computation of their *Matveev complexity* and geometry: see [8] and [9] for the orientable case and [4], [5] and [1] for the non-orientable one.

[1] We expect to succeed in significantly extending crystallization catalogues, both in dimension three and in dimension four, by optimizing the code and by exploiting high-powered computers, in virtue of the Italian Supercomputing Resource Allocation (ISCRA) project *"Cataloguing PL-manifolds in dimension 3 and 4 via crystallization theory"*, supported by CINECA.

Experimental data from these catalogues also yield interesting information in order to compare Matveev complexity with the so-called *gem-complexity* of a closed 3-manifold M, which involves the minimum order of a crystallization of M [2].[2]

As far as dimension 4 is concerned, the generation of manifolds catalogues implies the previous generation of all gems (not necessarily crystallizations) representing 3-dimensional spheres up to a fixed order; moreover, suitable sequences of combinatorial moves realizing the PL-classification of the represented 4-manifolds have to be chosen and implemented (see [3] and [17]).

The initial segment of 4-dimensional crystallizations catalogue allows to:

- *characterize* $\mathbb{S}^4$ *(resp.* $\mathbb{CP}^2$*) (resp.* $\mathbb{S}^1 \times \mathbb{S}^3$ *and* $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^3$*) among closed* 4*-manifolds by means of gem-complexity* 0 *(resp.* 3*) (resp.* 4*);*
- *check that no other closed handle-free* 4*-manifold exists with gem-complexity at most* 5*;*
- *check that* $\mathbb{RP}^4$ *has gem-complexity* 7 *and no other closed non-orientable handle-free* 4*-manifold exists with gem-complexity at most* 9*;*[3]
- *check that the manifolds* $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$ *and* $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$ *have gem-complexity* 6 *and any other closed orientable handle-free* 4*-manifold with gem-complexity* k, $6 \leq k \leq 9$, *is TOP-homeomorphic to one of them.*

Note that the PL-classification of the elements of our catalogue might provide interesting examples of different PL 4-manifolds triangulating the same topological 4-manifold. In fact, known properties of crystallizations, combined with the up-to-date topological classification of simply connected PL 4-manifolds (see [15], [14] and [16]), allow to prove that:
If M^4 *is a simply connected closed PL* 4*-manifold with gem-complexity* $k \leq 65$, *then* M^4 *is TOP-homeomorphic to either* $(\#_r \mathbb{CP}^2) \# (\#_{r'} - \mathbb{CP}^2)$ *with* $r + r' = \beta_2(M^4)$ *or* $\#_s(\mathbb{S}^2 \times \mathbb{S}^2)$ *with* $s = \frac{1}{2}\beta_2(M^4)$.

3. Complexity estimations

By making use of the strong connection existing in dimension 3 between gems and Heegaard diagrams, a 3-manifold invariant based on crystallization theory – called *GM-complexity* – has been introduced and proved to be an upper bound for the Matveev complexity of each compact 3-manifold (see [5], [6] and [8] for the closed case and [10] for the boundary case).

Experimental results concerning 3-manifolds admitting a crystallization with “few” vertices (namely less than 32), suggests the sharpness of this bound for all closed 3-manifolds.

The notion of *GM*-complexity, combined with the widely investigated relationships between crystallization theory and other representation methods for 3-manifolds, has allowed to obtain direct estimations of the Matveev complexity for

[2]The *gem-complexity* of a closed n-manifold M^n is the non-negative integer $k(M^n) = p - 1$, $2p$ being the minimum order of a crystallization of M^n.

[3]Actually, the standard order 16 crystallization of $\mathbb{RP}^4$ turns out to be the unique non-bipartite 5-coloured graph, within the catalogue of all rigid crystallizations lacking in dipoles up to 20 vertices.

several classes of manifolds, significantly improving former results: this happens, in particular, for two-fold branched coverings of $\mathbb{S}^3$, for three-fold simple branched coverings of $\mathbb{S}^3$, and for 3-manifolds obtained by Dehn surgery on framed links in $\mathbb{S}^3$ (see [6]).

Moreover, GM-complexity has been proved to coincide with the so called *modified Heegaard complexity*, another 3-manifold invariant introduced in [13] (by making use of *generalized Heegaard diagrams*) as an approach to Matveev complexity computation: see [11].

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Triangulations of Hyperbolic 3-Manifolds Admitting Strict Angle Structures

Henry Segerman
(joint work with Craig D. Hodgson and J. Hyam Rubinstein)

It is conjectured that every hyperbolic 3-manifold with torus boundary components has a decomposition into positive volume ideal hyperbolic tetrahedra (a *geometric* triangulation). Such a decomposition can be very useful and has been studied by many authors, starting with Thurston [6], who introduced his *gluing equations*, describing conditions on shapes of positive volume ideal hyperbolic tetrahedra so that they fit together properly in a geometric triangulation.

Epstein and Penner [1] showed that every cusped (i.e. non-compact with finite volume) hyperbolic 3-manifold has a canonical decomposition into convex ideal polyhedra. In many cases (for example punctured torus bundles and 2-bridge knot complements, see Guéritaud and Futer [2]) the polyhedra of this canonical decomposition are tetrahedra, and we get a geometric triangulation. However, in general the polyhedra may be more complicated than tetrahedra. The obvious approach to try to get a geometric ideal triangulation is to subdivide the polyhedra into tetrahedra. The difficulty is that the subdivision induces triangulations of the polygonal faces of the polyhedra, and these triangulations may not be consistent with each other where two polygonal faces are glued to each other. This can be fixed by inserting flat hyperbolic tetrahedra in between the two polyhedra, building a layered triangulation on the polygon that bridges between the two triangulations. The cost paid is the addition of the flat tetrahedra, and the resulting triangulation is not geometric.

Experimental evidence from SnapPea [8] supports the conjecture that every cusped hyperbolic 3-manifold has a geometric triangulation. Wada, Yamashita and Yoshida [7, 9] have proved the existence of such triangulations given certain combinatorial conditions on the polyhedral decomposition, and Luo, Schleimer and Tillmann [4] show that such triangulations exist virtually, but the general problem remains unsolved.

We investigated an easier problem, that of finding an ideal triangulation with a *strict angle structure*, which can be thought of as a solution to the rotational part of Thurston's gluing equations. To be precise, a *generalised angle structure* is an assignment of real numbers ("angles") to the edges of each tetrahedron so that opposite edges within each tetrahedron get the same angle, the sum of the six angles within each tetrahedron is 2π, and the sum of the angles around each edge of the triangulation (after gluing the tetrahedra together) is 2π. A *semi-angle structure* has all angles in $[0, \pi]$, and a *strict angle structure* has all angles in $(0, \pi)$. The space of semi-angle structures is then a convex polytope.

The existence of a strict angle structure on a triangulation is a necessary but not sufficient condition for the triangulation to be geometric. Casson showed that a manifold with an ideal triangulation that admits a strict angle structure also admits a complete hyperbolic structure, although this is an existence result that uses

Thurston's hyperbolization theorem and so is not constructive. However, a strict angle structure determines an ideal hyperbolic shape for each tetrahedron, and so a volume for each tetrahedron. Using this, one can define a volume functional on the space of strict angle structures by adding up the volumes of the tetrahedra. The volume functional extends continuously to the semi-angle structure polytope, and is concave down on this polytope. Casson and Rivin showed that if the maximum of the volume functional is achieved at a strict angle structure (rather than on the boundary of the semi-angle structure polytope), then the corresponding ideal hyperbolic tetrahedra fit together to give a geometric triangulation.

This then gives a possible strategy to finding a geometric triangulation: we find a triangulation that admits a strict angle structure, and then hope that the volume functional is maximised at a strict angle structure.

Our main result is:

Theorem. *Assume that M is a cusped hyperbolic 3-manifold homeomorphic to the interior of a compact 3-manifold $\overline{M}$ with torus or Klein bottle boundary components. If $H_1(\overline{M};\mathbb{Z}_2) \to H_1(\overline{M},\partial\overline{M};\mathbb{Z}_2)$ is the zero map then M has an ideal triangulation with a strict angle structure.*

Corollary. *If M is a hyperbolic link complement in S^3, then M admits an ideal triangulation with a strict angle structure.*

Proof. For a link $L \subset S^3$, the peripheral elements generate $H_1(\overline{M})$, where $\overline{M}$ is the complement of an open regular neighbourhood of L in S^3. This can be seen using a Mayer-Vietoris sequence, or just by observing that if we kill all of the meridian curves by filling in disks then we obtain S^3 minus a number of 3-balls, which has zero first homology. Therefore, the map to $H_1(\overline{M},\partial\overline{M})$ is the zero map. □

Unfortunately, the triangulations we find are not generally geometric.

The idea of the construction is similar to the outline above of a method to find an ideal triangulation from the Epstein-Penner polyhedral decomposition: we carefully choose a subdivision of the polyhedra into ideal tetrahedra, using a "coning" procedure, and then insert flat tetrahedra to bridge between the identified faces of polyhedra that do not have matching induced triangulations. This gives a triangulation with a natural semi-angle structure (which is not a strict angle structure because of the inserted flat tetrahedra). One approach to the result would be to try to deform the semi-angle structure into a strict angle structure, opening out the flat tetrahedra so that each one has positive volume. However, instead of trying to deform the angle structure directly, we use work of Kang and Rubinstein [3] and Luo and Tillmann [5] which tells us that a strict angle structure exists if and only if certain "vertical" normal surface classes in the triangulation do not exist.

Our construction gives us a large amount of control over how a vertical normal class can be arranged in the polyhedra and in the bridge regions. It can only have non-zero quadrilateral coordinates in the bridge regions, where we inserted flat tetrahedra, and in fact only in bridge regions at polygonal faces of the polyhedra with either 4 or 6 sides. We proceed by replacing the quadrilaterals in the bridge regions with certain twisted disk surface parts to convert the normal surface solution class into an embedded closed surface. A parity argument shows that a fundamental, non-peripheral component of the surface must have an odd number of twisted disk parts in some bridge region. Our coning procedure for subdividing the polyhedra can be chosen so that we can construct a loop that passes through this bridge region and no others, and this loop gives a contradiction to the homology condition, ruling out the possibility of the vertical normal surface class and thereby proving the theorem.

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Structure of 0-Efficient or Minimal Triangulations

Stephan Tillmann

(joint work with William Jaco and J. Hyam Rubinstein)

1. Complexity of 3-manifolds

In this report, all 3-manifolds are assumed to be closed, irreducible, orientable and connected. The *complexity* of the 3-manifold M, denoted $c(M)$, is the minimum number of tetrahedra in a (pseudo-simplicial) triangulation of M. This number agrees with the complexity defined by Matveev [8] unless the manifold is S^3, $\mathbb{R}P^3$ or $L(3,1)$. It follows from the definition that the complexity is known for all closed, irreducible manifolds which appear in certain computer generated censuses. The largest published census of closed orientable 3-manifolds at the time of writing is due to Matveev [12] and goes up to complexity 12, and Matveev [13] has announced the completion of the census of complexity 13. Some interesting facts about the census are the following:

(1) If $c(M) \leq 5$, then M is spherical. The manifolds of complexity one are $\mathbb{S}^3$, $L(4,1)$ and $L(5,2)$. The Poincaré homology sphere has complexity 5.
(2) If $c(M) \leq 8$, then M is a graph manifold.
(3) If M is euclidean, then $c(M) = 6$.
(4) If M is hyperbolic, then $c(M) \geq 9$. There are four hyperbolic manifolds of complexity 9, including the Fomenko–Matveev–Weeks's manifold.

Matveev uses spines to represent 3-manifolds. Independently, Burton [1] has produced a census of all minimal triangulations of the manifolds of complexity up to 11. These censuses of minimal spines or minimal triangulations are as useful as the tables of small knots and links for low-dimensional topologists. However, hyperbolic examples only appear from complexity 9 and are sparse amongst these low complexity manifolds (see [11] for an analysis). Moreover, determining the complexity of a given manifold is in general difficult. For instance, for the complexity of the Weber–Seifert dodecahedral space we have a lower bound of 14 due to the latest census [13], and an upper bound of 23 due to 3 inequivalent triangulations [2], but the exact value is not known.

2. Towards vertical sections of the census

Given the difficulty of determining the complexity of a 3-manifold, one is often content to find upper and lower bounds. Whilst an upper bound arises from the presentation of a manifold via a spine, a Heegaard splitting or a triangulation, Matveev [9] states that the problem of finding lower bounds is quite difficult. Lower bounds using homology groups or the fundamental group are given by Matveev and Pervova [14], and lower bounds using hyperbolic volume are given by Matveev, Petronio and Vesnin [15]. These bounds are only known to be sharp for a few examples, and often the known census will provide a better lower bound for a given manifold. For some classes of manifolds, one has natural spines or triangulations that lead to a conjectured minimal complexity:

Conjecture (Matveev, Jaco–Rubinstein)**.** *The complexity of the lens space $L(p,q)$ is $E(p,q)-3$, where $(p,q)=1$, $p>q>0$, $p>3$, and $E(p,q)$ is the sum, $\sum n_i$, of the "partial denominators" in the continued fractions expansion of p/q :*

$$\frac{p}{q} = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}}} = [n_0, n_1, n_2, n_3, \dots].$$

In joint work with Jaco and Rubinstein, this conjecture was verified for an infinite family of lens spaces with even fundamental group (see [4] for an explicit description of those lens spaces). This is the first infinite family of closed 3-manifolds for which the complexity is known. Using covering spaces (see [5] and

Sec. 4), we were able to boot-strap this result to produce more families of minimal triangulations, and in particular obtained the following result about *the* census of *all* minimal triangulations:

(5) For all $n \in \mathbb{N}$ there exist at least two spherical manifolds of complexity n, which have unique minimal triangulations.

A guiding principle in our approach has been to try to not only determine the complexity of a manifold, but also to know all minimal triangulations realising it. The combinatorial structure of a minimal triangulation is governed by 0-efficiency [3] and low degree edges [4]. From this one can extrapolate building blocks for minimal triangulations. Understanding how they fit together under extra constraints on the manifold is fundamental in our work. We hope that one can effectively understand *vertical sections* of the census which pick up a finite cover of every manifold. Especially with view towards infinite families of minimal triangulations of closed hyperbolic 3-manifolds, this seems to be the most promising approach to date as we still have little insight into additional structure of minimal triangulations of closed hyperbolic 3-manifolds.

3. Bounds from Thurston norm

Let M be a closed, orientable, irreducible, connected 3-manifold, and let S be a properly embedded surface dual to a given $\varphi \in H^1(M;\mathbb{Z}_2)$. An analogue of Thurston's norm [16] can be defined as follows. If S is connected, let

$$\chi_-(S) = \max\{0, -\chi(S)\},$$

and otherwise let

$$\chi_-(S) = \sum_{S_i \subset S} \max\{0, -\chi(S_i)\},$$

where the sum is taken over all connected components of S. Note that S_i is not necessarily orientable. Define:

$$|| \ \varphi \ || = \min\{\chi_-(S) \mid S \text{ dual to } \varphi\}.$$

The surface S dual to $\varphi \in H^1(M;\mathbb{Z}_2)$ is said to be $\mathbb{Z}_2$-*taut* if no component of S is a sphere and $\chi(S) = -|| \ \varphi \ ||$. As in [16], one observes that every component of a $\mathbb{Z}_2$-taut surface is non-separating and geometrically incompressible.

Theorem 1 (Thurston norm bounds complexity)**.** *Let M be a closed, orientable, irreducible, atoroidal, connected 3-manifold with triangulation $\mathcal{T}$, and denote by $|\mathcal{T}|$ the number of tetrahedra. If $H \leq H^1(M;\mathbb{Z}_2)$ is a subgroup of rank two, then:*

$$|\mathcal{T}| \geq 2 + \sum_{0 \neq \varphi \in H} || \ \varphi \ ||.$$

A characterisation of triangulations realising the above lower bound is given in the next result. Let $\mathcal{T}$ be a triangulation of M having a single vertex. Place three quadrilateral discs in each tetrahedron, one of each type, such that the result is a (possibly branched immersed) normal surface. This surface is denoted Q and called the *canonical quadrilateral surface.* Suppose Q is the union of three embedded normal surfaces. Then each of them meets each tetrahedron in a single quadrilateral disc and is hence a one-sided Heegaard splitting surface. It defines a dual $\mathbb{Z}_2$-cohomology class and $H^1(M;\mathbb{Z}_2)$ has rank at least two.

Theorem 2. *Let M be a closed, orientable, irreducible, atoroidal, connected 3-manifold with triangulation $\mathcal{T}$. Let $H \leq H^1(M;\mathbb{Z}_2)$ be a subgroup of rank two. Then the following two statements are equivalent.*

(1) We have
$$|\mathcal{T}| = 2 + \sum_{0\neq\varphi\in H} \| \varphi \|.$$

(2) The triangulation has a single vertex and the canonical quadrilateral surface is the union of three $\mathbb{Z}_2$-taut surfaces representing the non-trivial elements of H.

Note that (1) implies that $\mathcal{T}$ is minimal by Theorem 1. Moreover, (2) implies that each non-trivial element of H has a $\mathbb{Z}_2$-taut representative, which is a one-sided Heegaard splitting surface, and that each edge has even degree.

The proofs of the theorems are based on a refinement of the methods of our result for lens spaces [4]. The *twisted layered loop triangulation* of S^3/Q_{8k}, k any positive integer, satisfies the equivalent statements in Theorem 2.

4. Bounds from covering spaces

Suppose M is a 3-manifold having a connected double cover, $\widetilde{M}$. A one-vertex triangulation, $\mathcal{T}$, of M lifts to a 2-vertex triangulation, $\widetilde{\mathcal{T}}$, of $\widetilde{M}$. Because there are two vertices, the lifted triangulation will, in general, not be minimal. Choosing an edge, $\tilde{e}$, joining the two vertices, one may be able to *crush* $\tilde{e}$ and the tetrahedra incident with it to form a new one-vertex triangulation $\widetilde{\mathcal{T}}^*$ of $\widetilde{M}$. If $t(\tilde{e})$ denotes the number of tetrahedra incident with $\tilde{e}$, then $c(\widetilde{M}) \leq 2|\mathcal{T}| - t(\tilde{e})$. If the complexity of $\widetilde{M}$ is known, this line of argument can be used to show that a given triangulation of M must be minimal.

Theorem 3. *Let M be a closed, orientable, connected, irreducible 3-manifold, and suppose $\widetilde{M}$ is a connected double cover of M. If $c(M) \geq 2$, then it follows that $c(\widetilde{M}) \leq 2 \cdot c(M) - 3$.*

Theorem 4. *Let M be a closed, orientable, connected, irreducible 3-manifold, and suppose $\widetilde{M}$ is a connected double cover of M. If $c(\widetilde{M}) = 2 \cdot c(M) - 3$, then either*

(1) $\widetilde{M} = S^3$ and $M = \mathbb{R}P^3$, or

(2) $\widetilde{M} = L(2k,1)$ for some $k \geq 2$ and M has a unique minimal triangulation and is the lens space $L(4k, 2k-1)$ or the generalised quaternionic space S^3/Q_{4k}.

Fact (5) stated in Sec. 2 follows from this result and our previous work [4], which shows that $L(2k,1)$ has a unique minimal triangulation and complexity $2k-3$.

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Minutes of the Open Problem Session

1. **Frank Lutz.** Benedetti in his talk described an algorithm for finding random discrete Morse functions on a simplicial complex. Starting with the boundary of a polytope, we can ask how likely it is to get a perfect Morse function with just 2 critical points (min and max). (Equivalently, after removing one facet at random, how likely is it that a random collapsing sequence will collapse the rest to a point?) For the boundary of a 7-simplex the answer is not 100%, since we could get stuck on an 8-vertex dunce hat, but simulations show it is close to 100%. What can be said about the complicatedness (the expected complexity of a random Morse function) of boundaries of d-simplices? What happens under repeated barycentric subdivisions? (Note: since the meeting, Karim Adiprasito has answered the

last question, showing that the complicatedness of r-fold barycentric subdivisions grows at least exponentially in r.)

2. **Ben Burton.** There are finitely many closed 3-manifold triangulations with N tetrahedra. Is there a polynomial-time algorithm to pick one at random? What if we restrict to 3-spheres? Note that N. Dunfield and D. Thurston showed (2006, *Inventiones*) that the probability that a random gluing of tetrahedra produces a manifold goes to 0 for large N, making the obvious approach infeasible. Similarly, it is not clear whether applying random bistellar flips from a fixed starting sphere can work: the best known upper bound for the number of flips needed to connect two N-tetrahedron triangulations of S^3 is $e^{O(N^2)}$ by A. Mijatović (2003, *Pacific J. Math.*).

3. **Joseph Gubeladze.** Oda asked the following: do any two linear triangulations of the d-simplex have a common stellar refinement? (It is known that the triangulations can be connected by a sequence of stellar refinements and their inverses.)

4. **John Sullivan.** We have mainly discussed two types of triangulations of 3-manifolds: simplicial complexes, and the more general triangulations arising from face-pairings. Of course any triangulation becomes simplicial after two levels of barycentric subdivision. But are there closer relations between the complexity measures used in the two worlds? As mentioned in Swartz's talk, the Γ invariant ($\Gamma(\Delta) = h_2 - h_3 - 4h_4$ for a simplicial 3-manifold with boundary) is within a constant factor of the Matveev complexity, and is perhaps additive under connect sums.

5. **Alex Engström.** In a random graph model $G(n)$, a probability is assigned to each graph on n vertices. If the probability is invariant under permutation of vertices, and the restriction from n to m vertices gives the model $G(m)$, then it is an *exchangable random graph model.* These models are frequently studied in the emerging area of graph limits, but the foundational general theory is due to Aldous and Hoover long ago and the probability theory can be found in Kallenberg's textbooks. Using Boij–Söderberg theory, it was shown by Anderson Forsman and Engström that the homology of clique complexes of exchangable random graph models becomes concentrated as the number of vertices increases. The most rigid and basic class of exchangable random graphs models is the Erdős–Renyi model, and for those clique complexes Kahle presented sharp thresholds. For homotopy theory, Boij–Söderberg does not apply, and for another random model, Babson, Hoffman and Kahle showed that homology and homotopy vanishes at different points. How do the homotopy groups for clique complexes of exchangable random graphs behave? Are these random objects universal in any respect?

6. **Bruno Benedetti.** The total number of triangulations of S^3 with N tetrahedra is completely unknown, even asymptotically, but there are reasons to suspect that it is superexponential. Exponential upper bounds are known for many classes of 3-spheres, e.g. for shellable spheres or even locally constructible spheres.

Ehrenborg and Hachimori showed (2001, *Eur. J. Comb.*) that shellable spheres are tame, in the sense that no cycle of less than b edges forms a knot of bridge index b. Conjecture: there are only exponentially many tame spheres.

7. **Steve Klee.** A *PS-sphere* is a join of simplex boundaries; a *PS-ball* is the join of a simplex with a PS-sphere. A simplicial complex is called *PS-ear-decomposable* if $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$, where Σ_0 is a PS-sphere, the other Σ_j are PS-balls, and $\Sigma_j \cap (\Sigma_0 \cup \cdots \cup \Sigma_{j-1}) = \partial\Sigma_j$. M. Chari (1997, *Trans. AMS*) showed the complex of independent sets in any matroid is PS-ear-decomposable. Question: is it possible to extend a notion of shifting to PS-ear-decomposable complexes? This is interesting because the h-vector of a PS-decomposable complex is very easy to compute, and this might help give a new approach to solving Stanley's conjecture that the h-vector of a matroid complex is a pure $\mathcal{O}$-sequence.

8. **Carlo Petronio.** If M^3 is a cusped hyperbolic manifold, does it admit a triangulation into geometric ideal tetrahedra (with positive volume)? (The answer would be yes if we allowed flat tetrahedra of volume zero.) F. Luo, S. Schleimer and S. Tillmann (2008, *Proc. AMS*) have shown this is true virtually, i.e., for some finite cover. It also seems true experimentally. A related question is whether all minimal triangulations (having one vertex in each cusp and a minimal number of tetrahedra) are geometric. (Jaco suggests looking at the two known minimal triangulations of the $(-2, 3, 7)$ pretzel knot complement, one of which may be known to be non-geometric.)

9. **Paco Santos.** Let $f(d, n)$ denote the maximal (dual) diameter of a closed simplicial d-manifold with n vertices, $\bar{f}(d, n)$ the same for a manifold with boundary. Further let $\bar{f}_\psi(d, n)$ and $f_\psi(d, n)$ denote the same for pseudomanifolds with and without boundary.. Since $\bar{f}_\psi(2, n) = \Theta(n^2)$, via repeated joins we get $\bar{f}_\psi(d, n) \geq \Omega(n^{2d/3})$. On the other hand, the polynomial Hirsch conjecture says $f(d, n) \leq (n-1)(d+1)$. For surfaces ($d = 2$) we know $f(2, n) \leq n - 3$ and $\bar{f}(2, n) \leq 2n$. Is it true that for surfaces with boundary we have $\bar{f}(2, n) \leq n - 3$?

10. **Christian Haase.** Does every smooth lattice polytope have a unimodular triangulation? Smooth here means that the polytope is simple and the d edge vectors from any vertex to its neighbors have determinant ± 1, i.e., form a basis for $\mathbb{Z}^d$. Unimodular means that each simplex has volume $1/d!$ (the minimum possible). What about the normal smooth reflexive 7-polytope given by the following 11 facets:

$$\begin{aligned} x_2 - 2x_6 - x_7 &\leq 1, & 2x_6 - x_1 - x_2 &\leq 1, \\ 2x_1 - x_2 - x_3 - x_4 - x_5 + x_6 &\leq 1, & x_2 - x_6 &\leq 1, \end{aligned}$$

and $x_i \leq 1$ for $i = 1, \ldots, 7$? (It has 52 vertices and contains 6233 lattice points.) Does it have a (regular) unimodular triangulation? Is its toric ideal generated by quadrics?

11. **Ulrich Brehm.** Let T be a simplicial $(d-1)$-sphere linearly embeddable in $\mathbb{R}^d$. We say that a simplicial d-ball U with boundary $\partial U = T$ is *universal* for T if every linear embedding of T in $\mathbb{R}^d$ can be extended to a linear embedding of U. Is there a nice characterization of universal triangulations? Is there an algorithm to construct a universal ball U with prescribed boundary T? Is there a bound on how big U must be? (Note that Brehm has settled the case $d = 2$ but nothing is known in higher dimensions.)

12. **Karim Adiprasito.** Recently we have shown any manifold with a CAT(0) polyhedral metric admits a collapsible triangulation. Are there collapsible simplicial manifolds whose underlying space does not admit any CAT(0) polyhedral metric? Note that any collapsible PL triangulation is homeomorphic to a ball, by the work of Whitehead (1939), so examples would necessarily be non-PL. (Note that all collapsible triangulations are contractible, but B. Mazu1961, *Ann. Math.*) gave a contractible (PL) 4-manifold which is not homeomorphic to a ball and thus cannot admit any collapsible triangulation.)

13. **John Sullivan.** Consider the edge valences in a triangulation of a 3-manifold. Their average is always less than 6 and F. Luo and R. Stong (1993, *Trans. AMS*) showed the average can be less than 4.5 only for spherical manifolds. If all edges have valence less than 6, then curvature considerations show again the manifold is spherical. But N. Brady, J. McCammond and J. Meier showed (2004, *Proc. AMS*) that any orientable 3-manifold can be built with valences $\{4, 5, 6\}$. Can this be tightened, for instance to $\{5, 6\}$?

14. **Tamal Dey / Uli Wagner.** The maximum number of facets of a k-dimensional simplicial complex on n vertices is obviously $\binom{n}{k+1}$, which for fixed k is $\Theta(n^{k+1})$. Now let $f_k^d(n)$ denote the maximum over k-complexes that admit piecewise linear embeddings into $\mathbb{R}^d$. The cyclic polytopes show that

$$f_k^d(n) = \Omega\big(n^{\min(k+1, \lceil d/2 \rceil)}\big).$$

Conjecturally, this power is sharp. Indeed, a more precise conjecture is that the number $f_j(X)$ of j-faces of an embeddable complex X satisfies

$$f_j(X) \le C_{d,j} f_{\lceil d/2 \rceil - 1}(X),$$

for some constant $C_{d,j}$ depending only on d and j. Sarkaria claimed this in 1992, but his proof had errors; T. Dey and H. Edelsbrunner (1994, *Discr. Comp. Geom.*) proved it for $d = 3$. Note that by old theorems of E. van Kampen (1939, *Abh. Math. Sem. Uni. Hamburg*) and A. Flores (1933, *Ergeb. Math. Kolloq.*) a complex embeddable in dimension $d = 2k$ cannot contain the $(k+1)$-fold join of a discrete 3-point set as a subcomplex. By forbidden subhypergraph arguments of P. Erdős and M. Simonovits (1983, *Combinatorica*) it follows that such a complex has at most $n^{k+1-3^{-k}}$ faces. For instance for a 2-complex embedded in four-space we get

$$f_2^4(n) = O\big(n^{3-\frac{1}{9}}\big).$$

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