Multiscale Mapper: Topological Summarization via Codomain Covers

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Abstract

Summarizing topological information from datasets and maps defined on them is a central theme in topological data analysis. Mapper, a tool for such summarization, takes as input both a possibly high dimensional dataset and a map defined on the data, and produces a summary of the data by using a cover of the codomain of the map. This cover, via a pullback operation to the domain, produces a simplicial complex connecting the data points.

The resulting view of the data through a cover of the codomain offers flexibility in analyzing the data. However, it offers only a view at a fixed scale at which the cover is constructed. Inspired by the concept, we explore a notion of a tower of covers which induces a tower of simplicial complexes connected by simplicial maps, which we call *multiscale mapper*. We study the resulting structure, and design practical algorithms to compute its persistence diagrams efficiently. Specifically, when the domain is a simplicial complex and the map is a real-valued piecewise-linear function, the algorithm can compute the exact persistence diagram only from the 1-skeleton of the input complex. For general maps, we present a combinatorial version of the algorithm that acts only on *vertex sets* connected by the 1-skeleton graph, and this algorithm approximates the exact persistence diagram thanks to a stability result that we show to hold.

1 Introduction

Recent years have witnessed significant progress in applying topological ideas to analyzing complex and diverse data. Topological ideas can be particularly powerful in deriving a succinct and meaningful summary of input data. For example, the theory of *persistent homology* built upon [19, 20, 25] and other fundamental developments [3, 4, 6, 10, 11, 12, 15, 27], has provided a powerful and flexible framework for summarizing information of an input space or a scalar field into a much simpler object called the persistence diagram/barcode.

Modern data can be complex both in terms of the domain where they come from and in terms of properties/observations associated with them which are often modeled as maps. For example, we can have a set of patients, where each patient is associated with multiple biological markers, giving rise to a multivariate map from the space of patients to an image domain that may or may not be the Euclidean space. To this end, we

need to develop theoretically justified methods to analyze not only real-valued scalar fields, but also more complex maps defined on a given domain, such as multivariate, circle valued, sphere valued maps, etc. There has been interesting work in this direction, including multidimensional persistence [4, 5] and persistent homology for circular valued maps [3]. However, summarizing multivariate maps using these techniques appears to be challenging. Our approach takes a different direction, and is inspired by and based on the mapper methodology, recently introduced by Singh et al. in [26]. Taking an observation made in [26] regarding the behavior of Mapper under a change in the covers as a starting point, we study a multiscale version of mapper, which we will henceforth refer to as *multiscale mapper*, that is capable of producing a multiscale summary using a cover of the codomain at different scales.

Given a map $f : X \to Z$, Singh et al. proposed a novel concept to create a topological metaphor, called the *mapper*, for the structure behind f by pulling back a cover of the space Z to a cover on X through f. This mapper methodology is general: it can work with any (reasonably tame) continuous maps between two topological spaces, and it converts complex maps and covers of the target space, into simplicial complexes, which are much easier to manipulate computationally. It is also powerful and flexible– one can view the map f and a finite cover of the space Z as the lens through which the input data X is examined. By choosing different maps and covers, the resulting mapper representation captures different aspects of the input data. Indeed, the mapper methodology has been successfully applied to analyzing various types of data, see e.g. [22, 24], and it is a main component behind the data analytics platform developed by the company Ayasdi.

Contributions. Given an input map $f : X \to Z$ and a finite cover $\mathcal{U}$ of Z, the induced mapper $\mathrm{M}(\mathcal{U}, f)$ is a simplicial complex encoding the structure of f through the lens of Z. However, the simplicial complex $\mathrm{M}(\mathcal{U}, f)$ provides only *one* snapshot of X at a *fixed* scale as determined by the scale of the cover $\mathcal{U}$. Using the idea of persistence homology, we study the evolution of the mapper $\mathrm{M}(f, \mathcal{U}_\varepsilon)$ for a *tower of covers* $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_\varepsilon$ at multiple scales (indexed by ε).

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As an intuitive example, consider a real-valued function $f : X \to \mathbb{R}$, and a cover $\mathcal{U}_\varepsilon$ of $\mathbb{R}$ consisting of all possible intervals of length ε. Intuitively, as ε tends to 0, the corresponding Mapper $\mathrm{M}(f, \mathcal{U}_\varepsilon)$ approaches the Reeb graph of f. As ε increases, we look at the Reeb graph at coarser and coarser resolution. The multiscale mapper in this case roughly encodes this simplification process.

The *multiscale mapper* $\mathrm{MM}(\mathfrak{U}, f)$, which we formally define in §3, consists of a sequence of simplicial complexes connected with simplicial maps. Upon passing to homology with fields coefficients, the information in $\mathrm{MM}(\mathfrak{U}, f)$ can be further summarized in the corresponding persistence diagram. In other words, we can now summarize an input described by a multivariate (or circle/sphere valued) map into a single persistence diagram, much like in traditional persistence homology for real-valued functions.

In §4, we discuss the stability of the multiscale mapper, under changes in the input map and/or in the tower of covers $\mathfrak{U}$. Stability is a highly desirable property for a summary as it implies robustness to noise in data and in measurements. Interestingly, analogous to the case of homology versus persistence homology, mapper does not satisfy a stability property, whereas multiscale mapper does enjoy stability as we show in this paper.

To facilitate the broader usage of mapper and multiscale mapper as a data analysis tool, we develop efficient algorithms for computing and approximating mapper and multiscale mapper. In particular, even for piecewise-linear functions defined on a simplicial complex, the standard algorithm needs to determine for each simplex the subset (partial simplex) on which the function value falls within a certain range. In §5, we show that for such an input, it is sufficient to consider the restriction of the function to the 1-skeleton of the complex for computing the mapper and the multiscale mapper. Since the 1-skeleton (a graph) is typically much smaller in size than the full complex, this helps improving the time efficiency of computing the mapper and multiscale mapper outputs.

In §6, we consider the more general case of a map $f : X \to Z$ where X is a simplicial complex but Z is not necessarily real-valued. We show that there is an even simpler combinatorial version of the multiscale mapper, which only acts on *vertex sets* of X with connectivity given by the 1-skeleton graph of X [1]. The cost we pay here is that the resulting persistence diagram *approximates* (instead of computing exactly)

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[1] We note that a variant of this combinatorial version is what is currently used in the publicly available software for mapper in practice.

that of the standard multiscale mapper, and the tower of covers of Z needs to satisfy a "goodness" condition.

We remark that the main text of this paper focuses on providing an intuitive treatment of the multiscale mapper, its properties and the description of algorithms. Some technical details and proofs can be found in the Appendix.

2   Topological background and motivation

In this section we recall several facts about topological spaces and simplicial complexes [23]. Let K and L be two finite simplicial complexes over the vertex sets V_K and V_L, respectively. A set map $\phi : V_K \to V_L$ is a *simplicial map* if $\phi(\sigma) \in L$ for all $\sigma \in K$.

By an open cover of a topological space X we mean a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of open sets such that $\bigcup_{\alpha \in A} U_\alpha = X$. In this paper, whenever referring to an open cover, we will always assume that each U_α is path connected.

DEFINITION 1. (NERVE OF A COVER) *Given a finite cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a topological space X, we define the* nerve *of the cover $\mathcal{U}$ to be the simplicial complex $N(\mathcal{U})$ whose vertex set is the index set A, and where a subset $\{\alpha_0, \alpha_1, \ldots, \alpha_k\} \subseteq A$ spans a k-simplex in $N(\mathcal{U})$ if and only if $U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k} \neq \emptyset$.*

Suppose that we are given a topological space X equipped with a continuous map $f : X \to Z$ into a parameter space Z, where Z is equipped with an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ for some finite index set A. Since f is continuous, the sets $\{f^{-1}(U_\alpha), \alpha \in A\}$ form an open cover of X. For each α, we can now consider the decomposition of $f^{-1}(U_\alpha)$ into its path connected components, so we write $f^{-1}(U_\alpha) = \bigcup_{i=1}^{j_\alpha} V_{\alpha,i}$, where j_α is the number of path connected components $V_{\alpha,i}$'s in $f^{-1}(U_\alpha)$. We write $f^*(\mathcal{U})$ for the cover of X obtained this way from the cover $\mathcal{U}$ of Z and refer to it as the *pullback* cover of X induced by $\mathcal{U}$ via f.

Notice that there are pathological examples of f where $f^{-1}(U_\alpha)$ may shatter into infinitely many path components. This motivates us to consider *well-behaved* functions f: we require that for every path connected open set $U \subseteq Z$, the preimage $f^{-1}(U)$ has *finitely* many path connected components. An example of well-behaved functions is given by piecewise-linear real valued functions defined on finite simplicial complexes. It follows that for any well-behaved function $f : X \to Z$ and any finite open cover $\mathcal{U}$ of Z, the open cover $f^*(\mathcal{U})$ is also finite.

If not stated otherwise, all functions and all covers are assumed to be well-behaved and

finite, respectively. Consequently, all nerves of pullbacks of finite covers will also be finite.

DEFINITION 2. (MAPPER [26]) *Let X and Z be topological spaces and let $f : X \rightarrow Z$ be a well-behaved and continuous map. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a finite open cover of Z. The* mapper construction *arising from these data is defined to be the nerve simplicial complex of the pullback cover:* $M(\mathcal{U}, f) := N(f^*(\mathcal{U}))$.

REMARK 2.1. (FOR DEFINITION 2) *This construction is quite general. It encompasses both the Reeb graph and merge trees at once: consider X a topological space and $f : X \rightarrow \mathbb{R}$. Then, consider the following two options for $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, the other ingredient of the construction:*

- $U_\alpha = (-\infty, \alpha)$ *for $\alpha \in A = \mathbb{R}$. This corresponds to sublevel sets which in turn lead to merge trees.*

- $U_\alpha = (\alpha - \varepsilon, \alpha + \varepsilon)$ *for $\alpha \in A = \mathbb{R}$, for some fixed $\varepsilon > 0$. This corresponds to (ε-thick) level sets, which induce a relaxed notion of Reeb graphs.*

In these two examples, for simplicity of presentation, the set A is allowed to have infinite cardinality. Also, note one can take any *open cover of $\mathbb{R}$ in this definition. This may give rise to other constructions beyond merge trees or Reeb graphs. For instance, one may choose any point $r \in \mathbb{R}$ and let $U_\alpha = (r - \alpha, r + \alpha)$ for each $\alpha \in A = \mathbb{R}$ or other constructions.*

Maps between covers. If we have two covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ of a space X, a *map of covers* from $\mathcal{U}$ to $\mathcal{V}$ is a set map $\xi : A \rightarrow B$ so that $U_\alpha \subseteq V_{\xi(\alpha)}$ for all $\alpha \in A$. By an abuse of notation we also use ξ to indicate the map $\mathcal{U} \rightarrow \mathcal{V}$. Given such a map of covers, there is an induced simplicial map $N(\xi) : N(\mathcal{U}) \rightarrow N(\mathcal{V})$, given on vertices by the map ξ. Furthermore, if $\mathcal{U} \xrightarrow{\xi} \mathcal{V} \xrightarrow{\zeta} \mathcal{W}$ are three different covers of a topological space with the intervening maps of covers between them, then $N(\zeta \circ \xi) = N(\zeta) \circ N(\xi)$ as well.

The following simple lemma will be very useful later on (proof in Appendix A).

LEMMA 2.1. (INDUCED MAPS ARE CONTIGUOUS) *Let $\zeta, \xi : \mathcal{U} \rightarrow \mathcal{V}$ be any two maps of covers. Then, the simplicial maps $N(\zeta)$ and $N(\xi)$ are contiguous.*

Recall that two simplicial maps $h_1, h_2 : K \rightarrow L$ are *contiguous* if for all $\sigma \in K$ it holds that $h_1(\sigma) \cup h_2(\sigma) \in L$. In particular, contiguous maps induce identical maps at the homology level [23]. Lemma 2.1 implies that the map $H_*(N(\mathcal{U})) \rightarrow H_*(N(\mathcal{V}))$ thus induced can be deemed canonical.

Pullbacks. When we consider a space X equipped with a continuous map $f : X \rightarrow Z$ to a topological space Z, and we are given a map of covers $\xi : \mathcal{U} \rightarrow \mathcal{V}$ between covers of Z, there is a corresponding map of covers between the respective pullback covers of X: $f^*(\xi) : f^*(\mathcal{U}) \longrightarrow f^*(\mathcal{V})$.

Indeed, we only need to note that if $U \subseteq V$, then $f^{-1}(U) \subseteq f^{-1}(V)$, and therefore it is clear that each path connected component of $f^{-1}(U)$ is included in exactly one path connected component of $f^{-1}(V)$. More precisely, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, $\mathcal{V} = \{V_\beta\}_{\beta \in B}$, with $U_\alpha \subseteq V_{\xi(\alpha)}$ for $\alpha \in A$. Let $\widehat{U}_{\alpha,i}$, $i \in \{1, \ldots, n_\alpha\}$ denote the connected components of $f^{-1}(U_\alpha)$ and $\widehat{V}_{\beta,j}$, $j \in \{1, \ldots, m_\beta\}$ denote the connected components of $f^{-1}(V_\beta)$. Then, the map of covers $f^*(\xi)$ from $f^*(\mathcal{U})$ to $f^*(\mathcal{V})$ is given by requiring that each set $\widehat{U}_{\alpha,i}$ is sent to the *unique* set of the form $\widehat{V}_{\xi(\alpha),j}$ so that $\widehat{U}_{\alpha,i} \subseteq \widehat{V}_{\xi(\alpha),j}$.

Furthermore, if $\mathcal{U} \xrightarrow{\xi} \mathcal{V} \xrightarrow{\zeta} \mathcal{W}$ are three different covers of a topological space with the intervening maps of covers between them, then $f^*(\zeta \circ \xi) = f^*(\zeta) \circ f^*(\xi)$.

3 Multiscale Mapper

In the definition below, *objects* can be covers, simplicial complexes, or vector spaces.

DEFINITION 3. (TOWER) *A* tower $\mathfrak{W}$ *with resolution $r \in \mathbb{R}$ is any collection $\mathfrak{W} = \{\mathcal{W}_\varepsilon\}_{\varepsilon \geq r}$ of objects $\mathcal{W}_\varepsilon$ together with maps $w_{\varepsilon,\varepsilon'} : \mathcal{W}_\varepsilon \rightarrow \mathcal{W}_{\varepsilon'}$ so that $w_{\varepsilon,\varepsilon} = \text{id}$ and $w_{\varepsilon',\varepsilon''} \circ w_{\varepsilon,\varepsilon'} = w_{\varepsilon,\varepsilon''}$ for all $r \leq \varepsilon \leq \varepsilon' \leq \varepsilon''$. Sometimes we write $\mathfrak{W} = \{\mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon,\varepsilon'}} \mathcal{W}_{\varepsilon'}\}_{r \leq \varepsilon \leq \varepsilon'}$ to denote the collection with the maps. Given such a tower $\mathfrak{W}$,* res$(\mathfrak{W})$ *refers to its resolution.*

When $\mathfrak{W}$ is a collection of finite *covers equipped with maps of covers between them, we call it* a tower of covers. *When $\mathfrak{W}$ is a collection of* finite *simplicial complexes equipped with simplicial maps between them, we call it* a tower of simplicial complexes.

The notion of resolution, and the variable ε, intuitively specify the granularity of the covers and the simplicial complexes induced by them.

The pullback properties described at the end of §2 make it possible to take the pullback of a given tower of covers of a space via a given continuous function into another space, so that we obtain:

PROPOSITION 3.1. *Let $\mathfrak{U}$ be a tower of covers of Z and $f : X \rightarrow Z$ be a continuous function. Then, $f^*(\mathfrak{U})$ is a tower of covers of X.*

In general, given a tower of covers $\mathfrak{W}$ of a space X, the nerve of each cover in $\mathfrak{W}$ together with simplicial maps induced by each map of $\mathfrak{W}$ provides a tower of simplicial complexes which we denote by $N(\mathfrak{W})$.

DEFINITION 4. (MULTISCALE MAPPER) *Let X and Z be topological spaces and $f : X \to Z$ be a continuous map. Let $\mathfrak{U}$ be a tower of covers of Z. Then, the multiscale mapper is defined to be the tower of simplicial complexes defined by the nerve of the pullback:*

$$\mathrm{MM}(\mathfrak{U}, f) := N(f^*(\mathfrak{U})).$$

Consider for example a sequence $\mathrm{res}(\mathfrak{U}) \leq \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$ of n distinct real numbers. Then, the definition of multiscale mapper $\mathrm{MM}(\mathfrak{U}, f)$ gives rise to the following:

$$(3.1)\ \ N(f^*(\mathcal{U}_{\varepsilon_1})) \to N(f^*(\mathcal{U}_{\varepsilon_2})) \to \cdots \to N(f^*(\mathcal{U}_{\varepsilon_n}))$$

which is a sequence of simplicial complexes connected by simplicial maps. Applying to them the homology functor $\mathrm{H}_k(\cdot)$, $k = 0, 1, 2, \ldots$, with coefficients in a field, one obtains persistence modules [17]: tower of vector spaces connected by linear maps for which efficient persistence algorithm is known [15]:

$$(3.2)\ \ \ \ \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_1}))\big) \to \cdots \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_n}))\big).$$

More importantly, the information contained in a persistence module can be summarized by simple descriptors: its associated persistence diagrams. As pointed out in [6], a finiteness condition is required, but given our assumptions that the covers are finite and that the function f is well-behaved we obtain that the homology groups of all nerves have finite dimensions. Now one can summarize the persistence module with a finite persistent diagram for the sequence $\mathrm{MM}(\mathfrak{U}, f)$, denoted by $\mathrm{D_kMM}(\mathfrak{U}, f)$ for each dimension $k \in \mathbb{N}$ (see [17] for background on persistence diagrams).

4 Stability

To be useful in practice, the multiscale mapper should be stable against the perturbations in the maps and the covers. we show that such a stability is enjoyed by the multiscale mapper under some natural condition on the tower of covers. The notion of stability in the context of topological data analysis has been recently studied by many researchers, see e.g. [1, 2, 6, 7, 11]. In particular, Cohen-Steiner et al. [11] showed that persistence diagrams are *stable* in the *bottleneck distance* meaning that small perturbations to a map yield small variations in the computed persistence diagrams. In [6], stability is expressed directly at the (algebraic) level of persistence modules (3.2) via a quantitative structural condition called *interleaving* of pairs of persistence modules. Before we state our stability results, we identify compatible notions of interleaving for tower of covers and tower of simplicial complexes, as a way to measure the "closeness" between two tower of covers (or two tower of simplicial complexes).

4.1 Interleaving of towers of covers and simplicial complexes. In this section we consider towers of covers and simplicial complexes indexed over $\mathbb{R}$. In practice, we often have towers indexed by a discrete set in $\mathbb{R}$. Any such tower can be extended to a tower of covers (or simplicial complexes) indexed over $\mathbb{R}$ by taking the covers for any intermediate open interval $(\varepsilon, \varepsilon') \subset \mathbb{R}$ to be same as that at $\varepsilon \in \mathbb{R}$.

DEFINITION 5. (INTERLEAVING OF TOWERS OF COVERS) *Let $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}$ and $\mathfrak{V} = \{\mathcal{V}_\varepsilon\}$ be two towers of covers of a topological space X such that $\mathrm{res}(\mathfrak{U}) = \mathrm{res}(\mathfrak{V}) = r$. Given $\eta \geq 0$, we say that $\mathfrak{U}$ and $\mathfrak{V}$ are η-interleaved if one can find maps of covers $\zeta_\varepsilon : \mathcal{U}_\varepsilon \to \mathcal{V}_{\varepsilon+\eta}$ and $\xi_{\varepsilon'} : \mathcal{V}_{\varepsilon'} \to \mathcal{U}_{\varepsilon'+\eta}$ for all $\varepsilon, \varepsilon' \geq r$.*

PROPOSITION 4.1. *(i) If $\mathfrak{U}$ and $\mathfrak{V}$ are η_1-interleaved and $\mathfrak{V}$ and $\mathfrak{W}$ are η_2-interleaved, then, $\mathfrak{U}$ and $\mathfrak{W}$ are $(\eta_1 + \eta_2)$-interleaved. (ii) Let $f : X \to Z$ be a continuous function and $\mathfrak{U}$ and $\mathfrak{V}$ be two η-interleaved tower of covers of Z. Then, $f^*(\mathfrak{U})$ and $f^*(\mathfrak{V})$ are also η-interleaved.*

DEFINITION 6. (INTERLEAVING OF SIMPLICIAL TOWERS) *Let $\mathfrak{S} = \big\{ \mathcal{S}_\varepsilon \xrightarrow{s_{\varepsilon,\varepsilon'}} \mathcal{S}_{\varepsilon'} \big\}_{r \leq \varepsilon \leq \varepsilon'}$ and $\mathfrak{T} = \big\{ \mathcal{T}_\varepsilon \xrightarrow{t_{\varepsilon,\varepsilon'}} \mathcal{T}_{\varepsilon'} \big\}_{r \leq \varepsilon \leq \varepsilon'}$ be two towers of simplicial complexes where $\mathrm{res}(\mathfrak{S}) = \mathrm{res}(\mathfrak{T}) = r$. We say that they are $\eta \geq 0$ interleaved if for each $\varepsilon \geq r$ one can find simplicial maps $\varphi_\varepsilon : \mathcal{S}_\varepsilon \to \mathcal{T}_{\varepsilon+\eta}$ and $\psi_\varepsilon : \mathcal{T}_\varepsilon \to \mathcal{S}_{\varepsilon+\eta}$ so that:*

(i) for all $\varepsilon \geq r$, $\psi_{\varepsilon+\eta} \circ \varphi_\varepsilon$ and $s_{\varepsilon,\varepsilon+2\eta}$ are contiguous,

(ii) for all $\varepsilon \geq r$, $\varphi_{\varepsilon+\eta} \circ \psi_\varepsilon$ and $t_{\varepsilon,\varepsilon+2\eta}$ are contiguous,

(iii) for all $\varepsilon' \geq \varepsilon \geq r$, $\varphi_{\varepsilon'} \circ s_{\varepsilon,\varepsilon'}$ and $t_{\varepsilon+\eta,\varepsilon'+\eta} \circ \varphi_\varepsilon$ are contiguous,

(iv) for all $\varepsilon' \geq \varepsilon \geq r$, $s_{\varepsilon+\eta,\varepsilon'+\eta} \circ \psi_\varepsilon$ and $\psi_{\varepsilon'} \circ t_{\varepsilon,\varepsilon'}$ are contiguous.

These four conditions are summarized by requiring that the four diagrams below commute up to contiguity:

(4.3)

$$\begin{array}{ccc}
\mathcal{S}_{\varepsilon+\eta} & \xrightarrow{\;s_{\varepsilon+\eta,\varepsilon'+\eta}\;} & \mathcal{S}_{\varepsilon'+\eta} \\
\;\nearrow^{\psi_\varepsilon} & & \;\nearrow^{\psi_{\varepsilon'}} \\
\mathcal{T}_\varepsilon & \xrightarrow{\;t_{\varepsilon,\varepsilon'}\;} & \mathcal{T}_{\varepsilon'}
\end{array}$$

The proof of the following result is in Appendix B.

PROPOSITION 4.2. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be two η-interleaved towers of covers of X with $\mathrm{res}(\mathfrak{U}) = \mathrm{res}(\mathfrak{V})$. Then, $N(\mathfrak{U})$ and $N(\mathfrak{V})$ are also η-interleaved.*

From now on, for a finite tower of simplicial complexes $\mathfrak{S}$ and $k \in \mathbb{N}$, we denote by $\mathrm{D}_k\mathfrak{S}$ the k-th persistence diagram of $\mathfrak{S}$ with coefficients in a fixed field. Notice that, since contiguous maps induce identity maps at the homology level, applying the simplicial homology functor with coefficients in a field to diagrams such as those in (4.3) yields two persistence modules *strongly* interleaved in the sense of [6]. Thus, we have a stability result for $\mathrm{D}_k\mathrm{MM}(\mathfrak{U}, f)$ when f is kept fixed but the tower of covers $\mathfrak{U}$ is perturbed.

COROLLARY 4.1. *For $\eta \geq 0$, let $\mathfrak{U}$ and $\mathfrak{V}$ be two finite towers of covers of Z with $\mathrm{res}(\mathfrak{U}) = \mathrm{res}(\mathfrak{V}) > 0$. Let $f : X \to Z$ be well-behaved and $\mathfrak{U}$ and $\mathfrak{V}$ be η-interleaved. Then, $\mathrm{MM}(\mathfrak{U}, f)$ and $\mathrm{MM}(\mathfrak{V}, f)$ are η-interleaved. In particular, the bottleneck distance between the persistence diagrams $\mathrm{D}_k\mathrm{MM}(\mathfrak{U}, f)$ and $\mathrm{D}_k\mathrm{MM}(\mathfrak{V}, f)$ is at most η for all $k \in \mathbb{N}$.*

4.2 Stability of Multiscale Mapper. The *fixed* simplicial complex produced by the standard mapper may not admit a simple notion of stability. We elaborate this point by an example in Appendix C. In this section, we show that the multiscale mapper on the other hand exhibits a stability property against perturbations both in functions and in tower of covers (the latter of which already discussed in Corollary 4.1). However, to guarantee stability against changes in functions, it is necessary to restrict the multiscale mapper to a special class of towers of covers, called *(c,s)-good tower of covers*. We justify the necessity of considering such (c, s)-good tower of covers in Appendix D.1.

In what follows, we assume that the target compact topological space Z is endowed with a metric d_Z. For a subset $O \subset Z$, let $\mathrm{diam}(O)$ denote its *diameter*, that is, the number $\sup_{z,z' \in O} d_Z(z, z')$. For $\delta \geq 0$, let O^δ denote the set $\{z \in Z \mid d_Z(z, O) \leq \delta\}$.

DEFINITION 7. (GOOD TOWER OF COVERS) *Let $c \geq 1$ and $s > 0$. We say that a finite tower of covers $\mathfrak{W} = \{\mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon,\varepsilon'}} \mathcal{W}_{\varepsilon'}\}_{\varepsilon \leq \varepsilon'}$ for the compact metric space (Z, d_Z) is (c, s)-good if:*

1. *$\mathrm{res}(\mathfrak{W}) = s$, and $s \leq \mathrm{diam}(Z)$;*

2. *$\mathrm{diam}(W) \leq \varepsilon$ for all $W \in \mathcal{W}_\varepsilon$ and all $\varepsilon \geq s$; and*

3. *for all $O \subset Z$ with $\mathrm{diam}(O) \geq s$, there exists $W \in \mathcal{W}_{c \cdot \mathrm{diam}(O)}$ such that $W \supseteq O$.* [2]

Intuitively, in a (c, s)-good tower of covers, the parameter ε of a cover $\mathcal{W}_\varepsilon \in \mathfrak{W}$ is a scale parameter, which specifies an upper-bound for the *size / resolution* of cover elements in $\mathcal{W}_\varepsilon$. Condition-3 in the above definition requires that any set $O \subset Z$ should be covered by an element from a cover of $\mathfrak{W}$ whose resolution is comparable to the size (diameter) of O. See Appendix D for more discussions on the (c, s)-good tower of covers and its properties. In particular, in Appendix D.1, we show that such a tower of covers is somewhat necessary to obtain a stability result. The (c, s)-good tower of covers is also computationally feasible. In Appendix D.2 we provide an example of a construction of a tower of covers for compact metric spaces with bounded doubling dimension using ε-nets.

Now, we prove the main stability result, Theorem 4.2. In particular, in Section 4.3 we first inspect the impact of function perturbation, and we prove the general result in Section 4.4.

4.3 Stability against function perturbation. Our study of stability against function perturbation involves a reindexing of the involved towers of covers.

DEFINITION 8. (REINDEXING) *Let $\mathfrak{W} = \{\mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon,\varepsilon'}} \mathcal{W}_{\varepsilon'}\}_{r \leq \varepsilon \leq \varepsilon'}$ be a tower of covers of Z with $r = \mathrm{res}(\mathfrak{W}) > 0$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function. Consider the tower of covers $\mathrm{R}_\phi(\mathfrak{W}) := \{\mathcal{V}_\varepsilon \xrightarrow{v_{\varepsilon,\varepsilon'}} \mathcal{V}_{\varepsilon'}\}_{\phi(r) \leq \varepsilon \leq \varepsilon'}$ given by*

- *$\mathcal{V}_\varepsilon := \mathcal{W}_{\phi^{-1}(\varepsilon)}$ for each $\varepsilon \geq \phi(r)$ and*

- *$v_{\varepsilon,\varepsilon'} = w_{\phi^{-1}(\varepsilon), \phi^{-1}(\varepsilon')}$ for $\varepsilon, \varepsilon'$ such that $\phi(r) \leq \varepsilon \leq \varepsilon'$.*

We refer to $\mathrm{R}_\phi(\mathfrak{W})$ as a reindexed *tower.*

In our case, we will use the log function to reindex a tower of covers $\mathfrak{W}$. We also need the following definition in order to state the stability results.

DEFINITION 9. *Given a tower of covers $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}$ and $\varepsilon_0 \geq \mathrm{res}(\mathfrak{U})$, we define the ε_0-truncation of $\mathfrak{U}$ as the tower $\mathrm{Tr}_{\varepsilon_0}(\mathfrak{U}) := \{\mathcal{U}_\varepsilon\}_{\varepsilon_0 \leq \varepsilon}$. Observe that, by definition $\mathrm{res}(\mathrm{Tr}_{\varepsilon_0}(\mathfrak{U})) = \varepsilon_0$.*

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[2]This condition is related to the concept of Lebesgue number for covers of metric spaces where metric balls instead of sets with bounded diameter are used.

PROPOSITION 4.3. *Let X be a compact topological space, (Z, d_Z) be a compact path connected metric space, and $f, g : X \to Z$ be two continuous functions such that for some $\delta \geq 0$ one has that $\max_{x \in X} d_Z(f(x), g(x)) = \delta$. Let $\mathfrak{W}$ be any (c, s)-good tower of covers of Z. Let $\varepsilon_0 = \max(1, s)$. Then, the $\log \varepsilon_0$-truncations of $\mathrm{R}_{\log}(f^*(\mathfrak{W}))$ and $\mathrm{R}_{\log}(g^*(\mathfrak{W}))$ are $\log(2c \max(\delta, s) + c)$-interleaved.*

Proof. For notational convenience write $\eta := \log(2c \max(\delta, s) + c)$, $\{\mathcal{U}_t\} = \mathfrak{U} := f^*(\mathfrak{W})$, and $\{\mathcal{V}_t\} = \mathfrak{V} := g^*(\mathfrak{W})$. With regards to satisfying Definition 6 for $\mathrm{R}_{\log}(\mathfrak{U})$ and $\mathrm{R}_{\log}(\mathfrak{V})$, for each $\varepsilon \geq \log \varepsilon_0$ we need only exhibit maps of covers $\zeta_\varepsilon : \mathcal{U}_{\exp(\varepsilon)} \to \mathcal{V}_{\exp(\varepsilon + \eta)}$ and $\xi_\varepsilon : \mathcal{V}_{\exp(\varepsilon)} \to \mathcal{U}_{\exp(\varepsilon + \eta)}$. We first establish the following.

CLAIM 1. *For all $O \subset Z$, and all $\delta' \geq \delta$, $f^{-1}(O) \subseteq g^{-1}(O^{\delta'})$.*

Proof. Let $x \in f^{-1}(O)$, then $d_Z(f(x), O) = 0$. Thus, $d_Z(g(x), O) \leq d_Z(f(x), O) + d_Z(g(x), f(x)) \leq \delta$, which implies the claim.

Now, pick any $t \geq \varepsilon_0$, any $U \in \mathcal{U}_t$, and fix $\delta' := \max(\delta, s)$. Then, there exists $W \in \mathcal{W}_t$ such that $U \in \mathrm{cc}(f^{-1}(W))$. The claim implies that $f^{-1}(W) \subseteq g^{-1}(W^{\delta'})$. Since $\mathfrak{W}$ is a (c, s)-good cover of the connected space Z and $s \leq \max(\delta, s) \leq \mathrm{diam}(W^{\delta'}) \leq 2\delta' + t$, there exists at least one set $W' \in \mathcal{W}_{c(2\delta'+t)}$ such that $W^{\delta'} \subseteq W'$. This means that U is contained in some element of $\mathrm{cc}(g^{-1}(W'))$ where $W' \in \mathcal{W}_{c(2\delta'+t)}$. But, also, since $c(2\delta' + t) \leq c(2\delta' + 1)t$ for $t \geq \varepsilon_0 \geq 1$, there exists $W'' \in \mathcal{W}_{c(2\delta'+1)t}$ such that $W' \subseteq W''$. This implies that U is contained in some element of $\mathrm{cc}(g^{-1}(W''))$ where $W'' \in \mathcal{W}_{c(2\delta'+1)t}$. This process, when applied to all $U \in \mathcal{U}_t$, all $t \geq \varepsilon_0$, defines a map of covers $\widehat{\zeta}_t : \mathcal{U}_t \to \mathcal{V}_{(2c\delta'+c)t}$. Now, define for each $\varepsilon \geq \log(\varepsilon_0)$ the map $\zeta_\varepsilon := \widehat{\zeta}_{\exp(\varepsilon)}$ and notice that by construction this map has $\mathcal{U}_{\exp(\varepsilon)}$ as domain, and $\mathcal{V}_{(2\delta'c+c)\exp(\varepsilon)}$ as codomain. A similar observation produces for each $\varepsilon \geq \log(\varepsilon_0)$ a map of covers ξ_ε from $\mathcal{V}_{\exp(\varepsilon)}$ to $\mathcal{V}_{(2c\delta'+c)\exp(\varepsilon)}$.

Notice that for each $\varepsilon \geq \log(\varepsilon_0)$ one may write $(2c\delta'+c)\exp(\varepsilon) = \exp(\varepsilon + \eta)$. So we have in fact proved that $\log \varepsilon_0$-truncations of $\mathrm{R}_{\log}(\mathfrak{U})$ and $\mathrm{R}_{\log}(\mathfrak{V})$ are η-interleaved.

An application of Proposition 4.3 and the result in [6] is (proof in Appendix E):

COROLLARY 4.2. *Let $\mathfrak{W}$ be any (c, s)-good tower of covers of the compact connected metric space Z and let $f, g : X \to Z$ be any two well-behaved continuous functions such that $\max_{x \in X} d_Z(f(x), g(x)) = \delta$. Then, the bottleneck distance between the persistence diagrams*

$\mathrm{D}_k\mathrm{MM}(\mathrm{R}_{\log}(\mathfrak{W}), f)$ *and* $\mathrm{D}_k\mathrm{MM}(\mathrm{R}_{\log}(\mathfrak{W}), g)$ *is at most* $\log(2c \max(s, \delta) + c) + \max(0, \log \frac{1}{s})$ *for all $k \in \mathbb{N}$.*

4.4 Stability in general. We now consider the more general case which also allows changes in the tower of covers inducing the multiscale mapper.

THEOREM 4.1. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be any two η-interleaved (c, s)-good towers of covers of the compact connected metric space Z and let $f, g : X \to Z$ be any two continuous well-behaved functions such that $\max_{x \in X} d_Z(f(x), g(x)) \leq \delta$. Then, for $\varepsilon_0 = \max(1, s)$, the $\log \varepsilon_0$-truncation of $\mathrm{R}_{\log}(f^*(\mathfrak{U}))$ and $\mathrm{R}_{\log}(g^*(\mathfrak{V}))$ are $\log(2c \max(s, \delta) + c + \eta)$-interleaved.*

Proof. Write $\mathrm{Tr}_{\varepsilon_0}(\mathfrak{V}) = \{\mathcal{V}_\varepsilon\}_{\varepsilon \geq \varepsilon_0}$ and $\mathrm{Tr}_{\varepsilon_0}(\mathfrak{U}) = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq \varepsilon_0}$. Denote $f^*(\mathfrak{V}) = \{\mathcal{V}_\varepsilon^f\}_{\varepsilon \geq \varepsilon_0}$, $g^*(\mathfrak{V}) = \{\mathcal{V}_\varepsilon^g\}_{\varepsilon \geq \varepsilon_0}$, and $f^*(\mathfrak{U}) = \{\mathcal{U}_\varepsilon^f\}_{\varepsilon \geq \varepsilon_0}$, $g^*(\mathfrak{U}) = \{\mathcal{U}_\varepsilon^g\}_{\varepsilon \geq \varepsilon_0}$. By following the argument and using the notation in Proposition 4.3, for each $\varepsilon \geq \varepsilon_0$ one can find maps of covers $\mathcal{U}_\varepsilon^g \to \mathcal{U}_{(2c\delta'+c)\varepsilon}^f$ and $\mathcal{V}_\varepsilon^f \to \mathcal{V}_{(2c\delta'+c)\varepsilon}^g$. Also, since $\mathfrak{U}^f$ and $\mathfrak{V}^f$ are η-interleaved, for each $\varepsilon \geq \varepsilon_0$ there are maps of covers $\mathcal{U}_\varepsilon^f \to \mathcal{V}_{\varepsilon+\eta}^f$ and $\mathcal{V}_\varepsilon^g \to \mathcal{U}_{\varepsilon+\eta}^g$. Then, for each $\varepsilon \geq \varepsilon_0$ one can form the following diagram

$$\mathcal{U}_\varepsilon^f \longrightarrow \mathcal{U}_{\varepsilon(2c\delta'+c)}^g \longrightarrow \mathcal{V}_{\varepsilon(2c\delta'+c)+\eta}^g \hookrightarrow \mathcal{V}_{\varepsilon(2c\delta'+c+\eta)}^g,$$

where the last step follows because since $\varepsilon \geq \varepsilon_0 \geq 1$, then we have that $\varepsilon(2c\delta'+c) + \eta \leq \varepsilon(2c\delta'+c+\eta)$. Thus, by composing the maps intervening in the diagram above we have constructed for any $\varepsilon \geq \varepsilon_0$ a map of covers $\mathcal{U}_\varepsilon^f \longrightarrow \mathcal{V}_{\varepsilon(2c\delta'+c+\eta)}^g$. In a similar manner one can construct a map of covers $\mathcal{V}_\varepsilon^g \longrightarrow \mathcal{U}_{\varepsilon(2c\delta'+c+\eta)}^f$ for each $\varepsilon \geq \varepsilon_0$.

This provides the maps $\mathcal{U}_{\exp(\varepsilon)}^f \to \mathcal{V}_{(2c\delta'+c+\eta)\exp(\varepsilon)}^g$ and $\mathcal{V}_{\exp(\varepsilon)}^g \to \mathcal{U}_{(2c\delta'+c+\eta)\exp(\varepsilon)}^f$ for each $\varepsilon \geq \varepsilon_0$. Since $(2c\delta'+c+\eta)\exp(\varepsilon) = \exp(\varepsilon + \log(2c\delta'+c+\eta))$, by reindexing by log we obtain that $\mathrm{R}_{\log}(f^*(\mathrm{Tr}_{\varepsilon_0}(\mathfrak{U})))$ and $\mathrm{R}_{\log}(g^*(\mathrm{Tr}_{\varepsilon_0}(\mathfrak{V})))$ are $\log(2c\delta'+c+\eta)$-interleaved.

Similar to the derivation of Corollary 4.2 from Proposition 4.3, we get the following result from Theorem 4.1.

THEOREM 4.2. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be any two η-interleaved, (c, s)-good tower of covers of the compact path connected metric space Z and let $f, g : X \to Z$ be any two well-behaved continuous functions such that $\max_{x \in X} d_Z(f(x), g(x)) \leq \delta$. Then, the bottleneck distance between $\mathrm{D}_k\mathrm{MM}(\mathrm{R}_{\log}(\mathfrak{U}), f)$ and $\mathrm{D}_k\mathrm{MM}(\mathrm{R}_{\log}(\mathfrak{V}), g)$ is bounded by $\log(2c \max(s, \delta) + c + \eta) + \max(0, \log \frac{1}{s})$ for all $k \in \mathbb{N}$.*

5 Exact Computation for PL-functions on simplicial domains

The stability result in Theorem 4.2 further motivates us to design efficient algorithms for constructing multiscale mapper or its approximation in practice. We justify the approximation algorithms using the stability results discussed above.

One of the most common types of input in practice is a real-valued piecewise-linear (PL) function $f : |K| \to \mathbb{R}$ defined on the underlying space $|K|$ of a simplicial complex K. That is, f is given at the vertex set $V(K)$ of K, and linearly interpolated within any other simplex $\sigma \in K$. In what follows, we consider this PL setting, and show that interestingly, if the input function satisfies a mild "minimum diameter" condition, then we can compute both mapper and multiscale mapper from simply the 1-skeleton (graph structure) of K. This makes the computation of the multiscale mapper from a PL function significantly faster and simpler as its time complexity depends on the size of the 1-skeleton of K, which is typically orders of magnitude smaller than the total number of simplices (such as triangles, tetrahedra, etc) in K.

Given a simplicial complex K, let K_1 denote the 1-skeleton of K: that is, K_1 contains the set of vertices and edges of K. Define $\tilde{f} : |K_1| \to \mathbb{R}$ to be the restriction of f to $|K_1|$; that is, $\tilde{f}$ is the PL function on $|K_1|$ induced by function values at vertices.

CONDITION 5.1. (MINIMUM DIAMETER CONDITION)
For a given tower of covers $\mathfrak{W}$ of a compact connected metric space (Z, d_Z), let $\kappa(\mathfrak{W}) := \inf\{\mathrm{diam}(W); W \in \mathcal{W} \in \mathfrak{W}\}$ denote the minimum diameter of any element of any cover of the tower $\mathfrak{W}$. Given a simplicial complex K with a function $f : |K| \to Z$ and a tower of covers $\mathfrak{W}$ of the metric space Z, we say that $(K, f, \mathfrak{W})$ satisfies the minimum diameter *condition if $\mathrm{diam}(f(\sigma)) \leq \kappa(\mathfrak{W})$ for every simplex $\sigma \in K$.*

In our case, f is a PL function, and thus satisfying the minimum diameter condition means that for every edge $e = (u, v) \in K_1$, $|f(u) - f(v)| \leq \kappa(\mathfrak{W})$. In what follows we assume that K is connected. We do not lose any generality by this assumption because the arguments below can be applied to each connected component of K.

DEFINITION 10. *Two towers of simplicial complexes $\mathfrak{S} = \left\{ \mathcal{S}_\varepsilon \xrightarrow{s_{\varepsilon,\varepsilon'}} \mathcal{S}_{\varepsilon'} \right\}$ and $\mathfrak{T} = \left\{ \mathcal{T}_\varepsilon \xrightarrow{t_{\varepsilon,\varepsilon'}} \mathcal{T}_{\varepsilon'} \right\}$ are isomorphic, denoted $\mathfrak{S} \cong \mathfrak{T}$, if $\mathrm{res}(\mathfrak{S}) = \mathrm{res}(\mathfrak{T})$, and there exist simplicial isomorphisms η_ε and $\eta_{\varepsilon'}$ such that the diagram below commutes for all $\mathrm{res}(\mathfrak{S}) \leq \varepsilon \leq \varepsilon'$.*

$$
\begin{array}{ccc}
\mathcal{S}_\varepsilon & \xrightarrow{s_{\varepsilon,\varepsilon'}} & \mathcal{S}_{\varepsilon'} \\
\eta_\varepsilon \| & & \| \eta_{\varepsilon'} \\
\mathcal{T}_\varepsilon & \xrightarrow{t_{\varepsilon,\varepsilon'}} & \mathcal{T}_{\varepsilon'}
\end{array}
$$

Our main result in this section is the following theorem which enables us to compute the mapper, multiscale mapper, as well as the persistence diagram for the multiscale mapper of a PL function f from its restriction $\tilde{f}$ to the 1-skeleton of the respective simplicial complex.

THEOREM 5.1. *Given a PL function $f : |K| \to \mathbb{R}$ and a tower of covers $\mathfrak{W}$ of the image of f with $(K, f, \mathfrak{W})$ satisfying the minimum diameter condition, one has $\mathrm{MM}(\mathfrak{W}, f) \cong \mathrm{MM}(\mathfrak{W}, \tilde{f})$.*

We show in Proposition 5.1 that the two mapper outputs $\mathrm{M}(\mathcal{W}, f)$ and $\mathrm{M}(\mathcal{W}, \tilde{f})$ are identical up to a relabeling of their vertices (hence simplicially isomorphic) for every $\mathcal{W} \in \mathfrak{W}$. Also, since the simplicial maps in the filtrations $\mathrm{MM}(\mathfrak{W}, f)$ and $\mathrm{MM}(\mathfrak{W}, \tilde{f})$ are induced by the pullback of the same tower of covers $\mathfrak{W}$, they are identical again up to the same relabeling of the vertices. This then establishes the theorem.

In what follows, for clarity of exposition, we use X and X_1 to denote the underlying space $|K|$ and $|K_1|$ of K and K_1, respectively. Also, we do not distinguish between a simplex $\sigma \in K$ and its image $|\sigma| \subseteq X$ and thus freely say $\sigma \subseteq X$ when it actually means that $|\sigma| \subseteq X$ for a simplex $\sigma \in K$.

PROPOSITION 5.1. *If $(K, f, \mathfrak{W})$ satisfies the minimum diameter condition, then for every $\mathcal{W} \in \mathfrak{W}$, $\mathrm{M}(\mathcal{W}, f)$ is identical to $\mathrm{M}(\mathcal{W}, \tilde{f})$ up to relabeling of the vertices.*

Proof. Let $\mathcal{U} = f^*(\mathcal{W})$ and $\tilde{\mathcal{U}} = \tilde{f}^*(\mathcal{W})$. By definition of $\tilde{f}$, each $\tilde{U} \in \tilde{\mathcal{U}}$ is a connected component of some $U \cap X_1$ for some $U \in \mathcal{U}$. In Proposition 5.2, we show that $U \cap X_1$ is connected for every $U \in \mathcal{U}$. Therefore, for every element $U \in \mathcal{U}$, there is a unique element $\tilde{U} = U \cap X_1$ in $\tilde{\mathcal{U}}$ and vice versa. Claim 2 finishes the proof.

CLAIM 2. *$\bigcap_{i=1}^{k} U_i \neq \emptyset$ if and only if $\bigcap_{i=1}^{k} \tilde{U}_i \neq \emptyset$.*

Proof. Clearly, if $\bigcap_{i=1}^{k} \tilde{U}_i \neq \emptyset$, then $\bigcap_{i=1}^{k} U_i \neq \emptyset$ because $\tilde{U}_i \subseteq U_i$. For the converse, assume $\bigcap_{i=1}^{k} U_i \neq \emptyset$ and pick any $x \in \bigcap_{i=1}^{k} U_i$. Let $\sigma \subseteq X$ be a simplex which contains x. Consider the level set $L = f^{-1}(f(x))$. Since f is PL, $L_\sigma = L \cap \sigma$ is connected and contains a point $y \in \sigma \cap X_1$. Then, $y \in U_i$ for each $i \in \{1, \ldots, k\}$ because x and y are connected by a path in $U_i \cap \sigma$ and $f(x) = f(y)$. Therefore, $y \in \bigcap_i U_i$. Since $y \in X_1$ it follows that $y \in \bigcap_i \tilde{U}_i$.

PROPOSITION 5.2. *If $(X, f, \mathfrak{W})$ satisfies the minimum diameter condition, then for every $\mathcal{W} \in \mathfrak{W}$ and every $U \in f^*(\mathcal{W})$, the set $U \cap X_1$ is connected.*

Proof. Fix $U \in f^*(\mathcal{W})$. If $U \cap X_1$ is not connected, let $C_1, \ldots, C_k$ denote its $k \geq 2$ connected components. First, we show that each C_i contains at least one vertex of X_1. Let $e = (u, v)$ be any edge of X_1 that intersects U. If both ends u and v lie outside U, then $|f(u) - f(v)| > |\max_U f - \min_U f| \geq \kappa(\mathfrak{W})$. But, this violates the minimum diameter condition. Thus, at least one vertex of e is contained in U. It immediately follows that C_i contains at least one vertex of X_1.

Let Δ be the set of all simplices $\sigma \subseteq X$ so that $\sigma \cap U \neq \emptyset$. Fix $\sigma \in \Delta$ and let x be any point in $\sigma \cap U$.

CLAIM 3. *There exists a point y in an edge of σ so that $f(x) = f(y)$.*

Proof. We first observe that $V^+(\sigma) := \{v \in V(\sigma) | f(v) \geq f(x)\}$ is non-empty. Otherwise $f(v) < f(x)$ for for all vertices $v \in \sigma$. In that case $f(x') < f(x)$ for all points $x' \in \sigma$ contradicting that x belongs to σ. Now, if $V^+(\sigma) = V(\sigma)$, there is an edge $e \subseteq \sigma$ so that $x \in e$ and taking $y = x$ serves the purpose. If $V^+(\sigma) \neq V(\sigma)$, there is a vertex $u \in V(\sigma)$ so that $f(u) < f(x)$. Taking a vertex v from the non-empty set $V^+(\sigma)$, we get an edge $e = (u, v)$ where $f(u) < f(x)$ and $f(v) \geq f(x)$. Since f is PL, it follows that e has a point y so that $f(y) = f(x)$.

Since σ contains an edge e that is intersected by U, it contains a vertex of e that is contained in U. This means every simplex $\sigma \in \Delta$ has a vertex contained in U. For each $i = 1, \ldots, k$ let $\Delta_i := \{\sigma \subseteq X \mid V(\sigma) \cap C_i \neq \emptyset.\}$ Since every simplex $\sigma \in \Delta$ has a vertex contained in U, we have $\Delta = \bigcup_i \Delta_i$. We argue that the sets $\Delta_1, \ldots, \Delta_k$ are disjoint from each other. Otherwise, there exist $i \neq j$ and a simplex σ with a vertex u in Δ_i and another vertex v in Δ_j. Then, the edge (u, v) must be in U because f is PL. But, this contradicts that C_i and C_j are disjoint. This establishes that each Δ_i is disjoint from each other and hence Δ is not connected contradicting that U is connected. Therefore, our initial assumption that $U \cap X_1$ is disconnected is wrong.

Real-valued functions on triangulable topological space. We note that, by using PL functions to approximate the continuous functions, the above result also leads to an approximation of the multiscale mapper for real-valued functions defined on triangulable topological spaces.

A PL function $f : |K| \to \mathbb{R}$ *δ-approximates* a continuous function $g : X \to \mathbb{R}$ defined on a topological space X if there exists a homeomorphism $h : X \to |K|$ such that for any point $y \in |K|$, we have that $|f(y) - g \circ h^{-1}(y)| \leq \delta$. The following result states that if a PL function δ-approximates a continuous real-valued function on X, then the persistence diagrams induced by the respective multiscale mappers are also close.

COROLLARY 5.1. *Let $f : |K| \to \mathbb{R}$ be a piecewise-linear function on K that δ-approximates a continuous function $g : X \to \mathbb{R}$. Let $\mathfrak{W}$ be a (c, s)-good tower of covers of $\mathbb{R}$. Then the bottleneck distance between the persistence diagrams $D_k MM(R_{\log}(\mathfrak{W}), f)$ and $D_k MM(R_{\log}(\mathfrak{W}), g)$ is at most $\log(2c \max(s, \delta) + c) + \max(0, \log \frac{1}{s})$ for all $k \in \mathbb{N}$.*

Proof. Let $\tilde{g} : |K| \to \mathbb{R}$ denote the push forward of $g : X \to \mathbb{R}$ by the homeomorphism $h : X \to |K|$. By the definition of δ-approximation, we know that $\|f - \tilde{g}\|_\infty \leq \delta$. Hence by Corollary 4.2, the bottleneck distance between the persistence diagrams $D_k MM(R_{\log}(\mathfrak{W}), f)$ and $D_k MM(R_{\log}(\mathfrak{W}), \tilde{g})$ is at most $\log(2c \max(s, \delta) + c) + \max(0, \log \frac{1}{s})$ for all $k \in \mathbb{N}$.

On the other hand, since h is homeomorphism, it is easy to verify that the pullback of $\mathfrak{W}$ via g and via $\tilde{g}$ induce isomorphic persistence modules: $H_k(MM(R_{\log}(\mathfrak{W}), g))) \cong H_k(MM(R_{\log}(\mathfrak{W}), \tilde{g})))$. Combining this with the discussion in previous paragraph, the claim then follows.

6 Approximating multiscale mapper for general maps

While results in the previous section concern real-valued PL-functions, we now provide a significant generalization for the case where f maps the underlying space of K into an arbitrary compact metric space Z. We present a "combinatorial" version of the (multiscale) mapper where each connected component of a pullback $f^{-1}(W)$ for any cover W in the cover of Z consists of only vertices of K. Hence, the construction of the Nerve complex for this modified (multiscale) mapper is purely combinatorial, simpler, and more efficient to implement. But we lose the "exactness", that is, in contrast with the guarantees provided by Theorem 5.1, the simpler combinatorial mapper only *approximates* the actual multiscale mapper at the homology level. Also, it requires a (c, s)-good tower of covers of Z. One more caveat is that the towers of simplicial complexes arising in this case do not interleave in the (strong) sense of Definition 6 but in a weaker sense. This limitation worsens the approximation result by a factor of 3.

In what follows, $cc(O)$ for a set O denotes the *set of all path connected components of O.*

Given a map $f : |K| \to Z$ defined on the underlying space $|K|$ of a simplicial complex K, to construct the
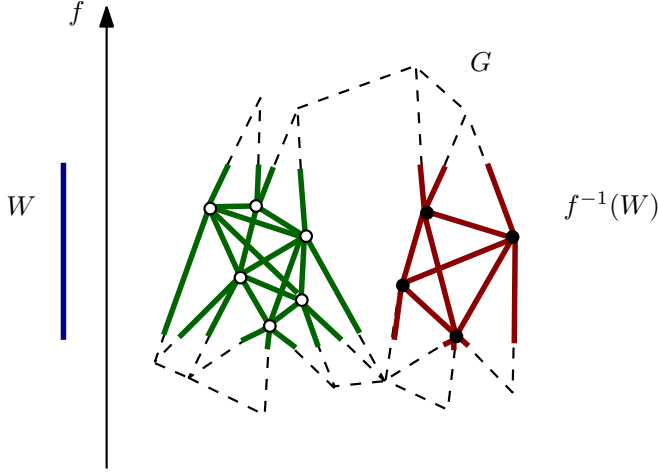
Figure 1: Partial thickened edges belong to the two connected components in $f^{-1}(W)$. Note that each set in $cc_G(f^{-1}(W))$ contains only the set of vertices of a component in $cc(f^{-1}(W))$.

mapper and multiscale mapper, one needs to compute the pullback cover $f^*(\mathcal{W})$ for a cover $\mathcal{W}$ of the compact metric space Z. Specifically, for any $W \in \mathcal{W}$ one needs to compute the pre-image $f^{-1}(W) \subset |K|$ and shatter it into connected components. Even in the setting adopted in §5, where we have a PL function $\tilde{f} : |K_1| \to \mathbb{R}$ defined on the 1-skeleton K_1 of K, the connected components in $cc(\tilde{f}^{-1}(W))$ may contain vertices, edges, and also *partial edges*: say for an edge $e \in K_1$, its intersection $e_W = e \cap f^{-1}(W) \subseteq e$, that is, $f(e_W) = f(e) \cap W$, is a partial edge. See Figure 1 for an example. In general for more complex maps, $\sigma \cap f^{-1}(W)$ for any k-simplex σ may be partial triangles, tetrahedra, etc., which can be nuisance for computations. The combinatorial version of mapper and multiscale mapper sidesteps this problem by ensuring that each connected component in the pullback $f^{-1}(W)$ consists of *only vertices of K*.

Combinatorial mapper and multiscale mapper.
Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose we are given a map $f : V(G) \to Z$ and a finite open cover $\mathcal{W} = \{W_\alpha\}_{\alpha \in A}$ of the metric space (Z, d_Z). For any $W_\alpha \in \mathcal{W}$, the pre-image $f^{-1}(W_\alpha)$ consists of a set of vertices which is shattered into subsets by the connectivity of the graph G. These subsets are taken as connected components. We now formalize this:

DEFINITION 11. *Given a set of vertices $O \subseteq V(G)$, the set of connected components of O induced by G, denoted by $cc_G(O)$, is the partition of O into a maximal subset of vertices connected in $G_O \subseteq G$, the subgraph spanned*

Algorithm 1 Combinatorial Multiscale Mapper

Input: $f : |K| \to Z$ given by $f_V : V(K) \to Z$, a tower of covers $\mathfrak{W} = \{\mathcal{W}_1, \ldots, \mathcal{W}_t\}$
Output: Persistence diagram $D_*(MM(\mathfrak{W}, f))$
    for $i = 1, \ldots, t$ **do**
        compute $V_W \subseteq V(K)$ where $f(V_W) = f(V(K)) \cap W$ and $\{V_W^j\}_j = cc_{K_1}(V_W)$, for $\forall W \in \mathcal{W}_i$;
        compute Nerve complex $N_i = N(\{V_W^j\}_{j,W})$.
    end for
    compute $D_*(\{N_i \to N_{i+1}, i \in [1, t-1]\})$.

---

by vertices in O. We refer to each such maximal subset of vertices as a *G-induced connected component* of O. We define $f^{*_G}(\mathcal{W})$, the *G-induced pull-back* via the function f, as the collection of all G-induced connected components $cc_G(f^{-1}(W_\alpha))$ for all $\alpha \in A$.

DEFINITION 12. *(G-induced multiscale mapper)* Similar to the mapper construction, we define the G-induced mapper $M^G(\mathcal{W}, f)$ as the nerve complex $N(f^{*_G}(\mathcal{W}))$.

    Given a tower of covers $\mathfrak{W} = \{\mathcal{W}_\varepsilon\}$ of Z, we define the G-induced multiscale mapper $MM^G(\mathfrak{W}, f)$ as the tower of G-induced nerve complexes $\{N(f^{*_G}(\mathcal{W}_\varepsilon)) \mid \mathcal{W}_\varepsilon \in \mathfrak{W}\}$.

Advantage of combinatorial multiscale mapper.
Given a map $f : |K| \to Z$ defined on the underlying space $|K|$ of a simplicial complex K, let $f_V : V(K) \to \mathbb{R}$ denote the restriction of f to the vertices of K. Consider the graph K_1 as providing connectivity information for vertices in $V(K)$. Given any tower of covers $\mathfrak{W}$ of the metric space Z, the K_1-induced multiscale mapper $MM^{K_1}(\mathfrak{W}, f_V)$ is called the *combinatorial multiscale mapper of f w.r.t. $\mathfrak{W}$*.

    A simple description of the computation of the combinatorial mapper is in Algorithm 1. For the simple PL example in Figure 1, $f^{-1}(W)$ contains two connected components, one consists of the set of white dots, while the other consists of the set of black dots. More generally, the construction of the pullback cover needs to inspect only the 1-skeleton K_1 of K, which is typically of significantly smaller size. Furthermore, the construction of the Nerve complex N_i as in Algorithm 1 is also much simpler: We simply remember, for each vertex $v \in V(K)$, the set I_v of ids of connected components $\{V_W^j\}_{j,W \in \mathcal{W}_i}$ which contain it. Any subset of I_v gives rise to a simplex in the Nerve complex N_i.

    Let $MM(\mathfrak{W}, f)$ denote the standard multiscale mapper as introduced in §3. Our main result in this section is that if $\mathfrak{W}$ is a (c, s)-good tower of covers of Z, then the resulting two towers of simplicial complexes, $MM(\mathfrak{W}, f)$ and $MM^{K_1}(\mathfrak{W}, f_V)$, interleave in a sense which is weaker

than that of Definition 6 but still admits a bounded distance between their respective persistence diagrams as a consequence of the weak interleaving result of [6]. This weaker setting only worsens the approximation by a factor of 3.

THEOREM 6.1. *Assume that (Z, d_Z) is a compact and connected metric space. Given a map $f : |K| \to Z$ defined on the underlying space of a simplicial complex K, let $f_V : \mathrm{V}(K) \to Z$ be the restriction of f to the vertex set $\mathrm{V}(K)$ of K. Given a (c, s)-good tower of covers $\mathfrak{W}$ of Z such that $(K, f, \mathfrak{W})$ satisfies the minimum diameter condition (cf. Condition 5.1), the bottleneck distance between the persistence diagrams $\mathrm{D}_k\mathrm{MM}\big(\mathrm{R}_{\log}(\mathfrak{W}), f\big)$ and $\mathrm{D}_k\mathrm{MM}^{K_1}\big(\mathrm{R}_{\log}(\mathfrak{W}), f_V\big)$ is at most $3\log(3c) + 3\max(0, \log\frac{1}{s})$ for all $k \in \mathbb{N}$.*

The remainder of this section is devoted to proving Theorem 6.1.

In what follows, the input tower of covers $\mathfrak{W} = \big\{\mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon,\varepsilon'}} \mathcal{W}_{\varepsilon'}\big\}_{\varepsilon \le \varepsilon'}$ is (c, s)-good, and we set $\rho = 3c$.

For each $\varepsilon \ge s$, let M_ε denote the the nerve complex $N(f^*(\mathcal{W}_\varepsilon))$ and $\mathrm{M}_\varepsilon^{K_1}$ denote the the combinatorial mapper $N(f_V^{*K_1}(\mathcal{W}_\varepsilon))$. Our goal is to show that there exist maps ϕ_ε and ν_ε so that diagram-(A) below commutes at the homology level, which then leads to a weak $\log \rho$-interleaving at log-scale of the persistence modules arising from computing the homologies of $\mathrm{MM}(\mathfrak{W}, f)$ and $\mathrm{MM}^{K_1}(\mathfrak{W}, f_V)$. This then proves Theorem 6.1.

(6.4)

$$(A) \qquad\qquad\qquad (B)$$

In the diagram above, the map $\theta_{\varepsilon,\varepsilon'} := N(f^*(w_{\varepsilon,\varepsilon'})) : \mathrm{M}_\varepsilon \to \mathrm{M}_{\varepsilon'}$ is the simplicial map induced by the pullback of the map of covers $w_{\varepsilon,\varepsilon'} : \mathcal{W}_\varepsilon \to \mathcal{W}_{\varepsilon'}$. Similarly, the map $\theta_{\varepsilon,\varepsilon'}^{K_1} := N(f_V^{*K_1}(w_{\varepsilon,\varepsilon'})) : \mathrm{M}_\varepsilon^{K_1} \to \mathrm{M}_{\varepsilon'}^{K_1}$ is the simplicial map induced by the pullback of the map of covers $w_{\varepsilon,\varepsilon'} : \mathcal{W}_\varepsilon \to \mathcal{W}_{\varepsilon'}$.

The maps ν_ε. Since connectivity in the 1-skeleton K_1 implies connectivity in the underlying space of K, there is a natural map from elements in $\mathrm{cc}_{K_1}(f_V^{-1}(W))$ to $\mathrm{cc}(f^{-1}(W))$, which in turn leads to a simplicial map $\nu_\varepsilon : \mathrm{M}_\varepsilon^{K_1} \to \mathrm{M}_\varepsilon$. The relatively simple details can be found in Appendix F.

Auxiliary maps of covers μ_ε. What remains is to define the maps ϕ_ε in diagram-(A). We first introduce the following map of covers: $\mu_\varepsilon : \mathcal{W}_\varepsilon \to \mathcal{W}_{\rho\varepsilon}$ for any $\varepsilon \ge s$. Specifically, given any $W \in \mathcal{W}_\varepsilon$ and $s \ge 0$, let $W^s = \{x \in Z \mid d_Z(x, W) \le s\}$. Since $\mathfrak{W}$ is (c, s)-good, there exists at least one set $W' \in \mathcal{W}_{\rho\varepsilon}$ such that $W^s \subseteq W'$. This is because for any $\varepsilon \ge s$:

$$s \le \mathrm{diam}(W^s) \le \mathrm{diam}(W) + 2s = \varepsilon + 2s \le 3\varepsilon,$$

so that then $c \cdot \mathrm{diam}(W^s) \le 3c\varepsilon = \rho\varepsilon$. We set $\mu_\varepsilon(W) = W'$: There may be multiple choices of sets in $\mathcal{W}_{\rho\varepsilon}$ that contains W, we pick an arbitrary but fixed one. Let $s_\varepsilon : \mathrm{M}_\varepsilon \to \mathrm{M}_{\rho\varepsilon}$ and $t_\varepsilon : \mathrm{M}_\varepsilon^{K_1} \to \mathrm{M}_{\rho\varepsilon}^{K_1}$ denote the simplicial map induced by the pullbacks of the cover map $\mu_\varepsilon : \mathcal{W}_\varepsilon \to \mathcal{W}_{\rho\varepsilon}$ via f and via f_V, respectively. In other words, $s_\varepsilon = N(f^*(\mu_\varepsilon))$ and $t_\varepsilon = N(f_V^{*K_1}(\mu_\varepsilon))$.

The maps ϕ_ε. We now define the map $\phi_\varepsilon : \mathrm{M}_\varepsilon \to \mathrm{M}_{\rho\varepsilon}^{K_1}$ with the help of diagram-(B) in (6.4). Fix $W \in \mathcal{W}_\varepsilon$. Given any set $U \in \mathrm{cc}(f^{-1}(W))$, write $\{V_\beta\}_{\beta \in A_U}$ for the preimage $\nu_\varepsilon^{-1}(U)$ of U under the vertex map ν_ε. Note that $V_\beta \subseteq U$ for any $\beta \in A_U$ and $\bigcup_{\beta \in A_U} V_\beta = \mathrm{V}(K) \cap U$.

We claim that $t_\varepsilon(V_\beta) = t_\varepsilon(V_{\beta'})$ for any $\beta, \beta' \in A_U$. Indeed, since V_β and $V_{\beta'}$ are contained in the path connected component U, let $\pi(x, y) \subseteq U$ be any path connecting a vertex $x \in V_\beta$ and $y \in V_{\beta'}$ in U. Let $\{\sigma_1, \ldots, \sigma_a\}$ be the collection of simplices that intersect $\pi(x, y)$. By the minimum diameter condition (cf. Condition 5.1) and the definition of the map μ_ε, we have that $f(\sigma_j) \subseteq W^s \subseteq \mu_\varepsilon(W) \in \mathcal{W}_{\rho\varepsilon}$, for any $j \in \{1, \ldots, a\}$. Hence $V_\beta \cup V_{\beta'} \cup \{\sigma_1, \ldots, \sigma_a\} \subseteq s_\varepsilon(U) \in \mathrm{cc}(f^{-1}(\mu_\varepsilon(W)))$. Furthermore, since the edges of simplices $\{\sigma_1, \ldots, \sigma_a\}$ connect $x \in V_\beta$ and $y \in V_{\beta'}$, vertices in V_β and $V_{\beta'}$ are thus connected by edges in $s_\varepsilon(U)$. That is, there exists a K_1-induced component (a subset of vertices of K) $\hat{V} \in \mathrm{cc}_{K_1}(f_V^{-1}(\mu_\varepsilon(W)))$ such that $V_\beta \cup V_{\beta'} \subseteq \hat{V}$ for any $\beta, \beta' \in A_U$. Hence, all V_βs with $\beta \in A_U$ have the same image $\hat{V}$ in $\mathrm{V}(\mathrm{M}_{\rho\varepsilon}^{K_1})$ under the map t_ε, and we set $\phi_\varepsilon(U) := \hat{V}$ to be this common image. The proof of the following result is in Appendix F.

LEMMA 6.1. *The vertex map ϕ_ε introduced above induces a simplicial map which we also denote by $\phi_\varepsilon : \mathrm{M}_\varepsilon \to \mathrm{M}_{\rho\varepsilon}^{K_1}$.*

Finally, by the construction of ϕ_ε, the lower triangle in diagram-(B) in (6.4) commutes. This fact, the commutativity of the square in diagram-(B), and the definition of ν_ε together imply that the top triangle commutes. Furthermore, Lemma 2.1 states the two maps of covers $f^*(w_{\varepsilon,\rho\varepsilon})$ and $f^*(\mu_\varepsilon)$ induce contiguous simplicial maps $\theta_{\varepsilon,\rho\varepsilon}$ and s_ε. Similarly, $\theta_{\varepsilon,\rho\varepsilon}^{K_1}$ is contiguous

to t_ε. Since contiguous maps induce the same map at the homology level, it then follows that diagram-(A) in (6.4) commutes at the homology level. At this point one could apply the strong interleaving result of [6] had the maps ϕ_ε been defined for any $M_\varepsilon \to M_{\varepsilon'}^{K_1}$. Nonetheless, the Weak Stability Theorem of [6] can be applied even in this case. This, together with an argument similar to the proof of Corollary 4.2, provides Theorem 6.1.

7 Discussion

In this paper, we proposed, as well as studied theoretical and computational aspects of *multiscale mapper,* a construction which produces a multiscale summary of a map on a domain using a cover of the codomain at different scales.

Given that in practice, hidden domains are often approximated by a set of discrete point samples, an important future direction will be to investigate how to approximate the multiscale mapper of a map $f : X \to Z$ on the metric space X from a finite set of samples P lying on or around X as well as function values $f : P \to Z$ at points in P. It will be particularly interesting to be able to handle noise both in point samples P and in the observed values of the function f.

We also note that the multiscale mapper framework can be potentially extended to a *zigzag*-tower of covers, which will further increase the information encoded in the resulting summary. It will be interesting to study the theoretical properties of such a *zigzag* version of mapper. Its stability however appears challenging, given that the theory of stability of zigzag persistence modules is much less developed than that for standard persistence modules. Another interesting question is to understand the continuous object that the mapper converges to as the scale of the cover tends to zero. In particular, does the mapper converge to the Reeb space [18]?

Finally, it seems of interest to understand the features of a topological space which are captured by Multiscale Mapper and its variants. Some related work in this direction in the context of Reeb graphs has been reported recently [13, 14].

References

[1] U. Bauer and M. Lesnick. Induced matchings of bar-codes and the algebraic stability of persistence. *Proc. 29th Annu. Sympos. Comput. Geom.* (2014), 355–364.

[2] P. Bubenik and J. A. Scott. Categorification of persistent homology. *Discrete & Computational Geometry* **51.3** (2014): 600–627.

[3] D. Burghelea and T. K. Dey. Topological persistence for circle-valued maps. *Discrete & Computational Geometry* **50.1** (2013), 69–98.

[4] G. Carlsson and A. Zomorodian. The theory of multidimensional persistence. *Discrete & Computational Geometry* **42.1** (2009), 71-93.

[5] A. Cerri, B. Di Fabio, M. Ferri, P. Frosini, and C. Landi. Betti numbers in multidimensional persistent homology are stable functions. *Mathematical Methods in the Applied Sciences.* **36** (2013), 1543– 1557.

[6] F. Chazal, D. Cohen-Steiner, M. Glisse, L. Guibas, and S. Oudot. Proximity of persistence modules and their diagrams. *Proc. 25th Annu. Sympos. Comput. Geom.(SoCG)* (2009), 237–246.

[7] F. Chazal, V. De Silva, M. Glisse, and S. Oudot. The structure and stability of persistence modules. ArXiv ArXiv:1207.3674 (2015).

[8] F. Chazal and J. Sun. Gromov-Hausdorff approximation of filament structure using Reeb-type graph. *Proc. 30th Annu. Sympos. Comput. Geom. (SoCG)* (2014), 491–500.

[9] C. Chen and M. Kerber. An output-sensitive algorithm for persistent homology. *Proc. 27th Annu. Sympos. Comput. Geom.(SoCG)* (2011), 207–216.

[10] G. Carlsson and V. de Silva. Zigzag persistence. *Found. Comput. Math.*, **10(4)**, 367–405, 2010.

[11] D. Cohen-Steiner, H. Edelsbrunner, and J. L. Harer. Stability of persistence diagrams. *Disc. Comput. Geom.* **37** (2007), 103-120.

[12] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Extending persistence using Poincaré and Lefschetz duality. *Found. of Comp. Math.* **9.1** (2009): 79-103.

[13] J. Curry. Personal communication.

[14] V. de Silva, E. Munch, and A. Patel. Categorified Reeb graphs. ArXiv preprint arXiv:1501.04147, (2015).

[15] T. K. Dey, F. Fan, and Y. Wang. Computing topological persistence for simplicial maps. *Proc. 30th Annu. Sympos. Comput. Geom. (SoCG)* (2014), 345–354.

[16] V. de Silva, D. Morozov, and M. Vejdemo-Johansson. Persistent cohomology and circular coordinates. *Discrete Comput. Geom.* **45 (4)** (2011), 737–759.

[17] H. Edelsbrunner and J. Harer. *Computational Topology: An Introduction.* Amer. Math. Soc., Providence, Rhode Island, 2009.

[18] H. Edelsbrunner, J. Harer, and A. K. Patel. Reeb spaces of piecewise linear mappings. In *Proc. 24th Annu. Sympos. Comput. Geom.(SoCG)* (2008), 242–250.

[19] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological persistence and simplification. *Discrete Comput. Geom.* **28** (2002), 511–533.

[20] P. Frosini. A distance for similarity classes of submanifolds of a Euclidean space, *Bulletin of the Australian*

Mathematical Society. **42(3)** (1990), 407–416.

[21] S. Har-Peled and M. Mendel. Fast construction of nets in low dimensional metrics, and their applications. *SIAM J. Comput..* **35(5)** (2006), 1148–1184.

[22] P.Y. Lum, G. Singh, A. Lehman, T. Ishkanov, M. Vejdemo-Johansson, M. Alagappan, J. Carlsson, and G. Carlsson. "Extracting insights from the shape of complex data using topology." *Scientific reports* **3** (2013).

[23] Munkres, J.R., *Topology*, Prentice-Hall, Inc., New Jersey, 2000.

[24] N., Monica, A. J. Levine, and G. Carlsson. Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival. *Proc. National Acad. Sci.* **108.17** (2011): 7265-7270.

[25] V. Robins. Towards computing homology from finite approximations. *Topology Proceedings.* **24(1)**. 1999.

[26] G. Singh, F. Mémoli, and G. Carlsson. Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition. *Sympos. Point Based Graphics*, 2007.

[27] A. Zomorodian and G. Carlsson. Computing persistent homology. *Discrete Comput. Geom.* **33** (2005), 249–274.

A Missing details from Section 2

Proof of lemma 2.1. Write $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$. Then, for all $\alpha \in A$ we have both

$$U_\alpha \subseteq V_{\zeta(\alpha)} \quad \text{and} \quad U_\alpha \subseteq V_{\xi(\alpha)}.$$

This means that $U_\alpha \subseteq V_{\zeta(\alpha)} \cap V_{\xi(\alpha)}$ for all $\alpha \in A$. Now take any $\sigma \in N(\mathcal{U})$. We need to prove that $\zeta(\sigma) \cup \xi(\sigma) \in N(\mathcal{V})$. For this write

$$\bigcap_{\beta \in \zeta(\sigma) \cup \xi(\sigma)} V_\beta = \left(\bigcap_{\alpha \in \sigma} V_{\zeta(\alpha)} \right) \cap \left(\bigcap_{\alpha \in \sigma} V_{\xi(\alpha)} \right)$$
$$= \bigcap_{\alpha \in \sigma} \left(V_{\zeta(\alpha)} \cap V_{\xi(\alpha)} \right)$$
$$\supseteq \bigcap_{\alpha \in \sigma} U_\alpha \neq \emptyset,$$

where the last step follows from assuming that $\sigma \in N(\mathcal{U})$.

B Missing details from Section 4.1.

Proof of Proposition 4.2. Let r denote the common resolution of $\mathfrak{U}$ and $\mathfrak{V}$. Write $\mathfrak{U} = \left\{ \mathcal{U}_\varepsilon \xrightarrow{u_{\varepsilon,\varepsilon'}} \mathcal{U}_{\varepsilon'} \right\}_{r \leq \varepsilon \leq \varepsilon'}$ and $\mathfrak{V} = \left\{ \mathcal{V}_\varepsilon \xrightarrow{v_{\varepsilon,\varepsilon'}} \mathcal{V}_{\varepsilon'} \right\}_{r \leq \varepsilon \leq \varepsilon'}$, and for each $\varepsilon \geq r$ let $\zeta_\varepsilon : \mathcal{U}_\varepsilon \to \mathcal{V}_{\varepsilon+\eta}$ and $\xi_\varepsilon : \mathcal{V}_\varepsilon \to \mathcal{U}_{\varepsilon+\eta}$ be given as in Definition 6. For each diagram in (4.3) one can consider a similar diagram at the level of covers involving covers of the form $\mathcal{U}_\varepsilon$ and $\mathcal{V}_\varepsilon$, and apply the nerve construction. This operation will yield diagrams identical to those in (4.3) where for each $\varepsilon \geq r$:

- $\mathcal{S}_\varepsilon := N(\mathcal{U}_\varepsilon)$, $\mathcal{T}_\varepsilon := N(\mathcal{V}_\varepsilon)$,

- $s_{\varepsilon,\varepsilon'} := N(u_{\varepsilon,\varepsilon'})$, for $r \leq \varepsilon \leq \varepsilon'$; $t_{\varepsilon,\varepsilon'} := N(v_{\varepsilon,\varepsilon'})$, for $r \leq \varepsilon \leq \varepsilon'$; $\varphi_\varepsilon := N(\zeta_\varepsilon)$, and $\psi_\varepsilon := N(\xi_\varepsilon)$.

To satisfy Definition 6, it remains to verify conditions (i) to (iv). We only verify (i), since the proof of the others follows the same arguments. For this, notice that both the composite map $\xi_{\varepsilon+\eta} \circ \zeta_\varepsilon$ and $u_{\varepsilon,\varepsilon+2\eta}$ are maps of covers from $\mathcal{U}_\varepsilon$ to $\mathcal{U}_{\varepsilon+2\eta}$. By Lemma 2.1 we then have that $N(\xi_{\varepsilon+\eta} \circ \zeta_\varepsilon)$ and $N(u_{\varepsilon,\varepsilon+2\eta}) = s_{\varepsilon,\varepsilon+2\eta}$ are contiguous. But, by the properties of the nerve construction $N(\xi_{\varepsilon+\eta} \circ \zeta_\varepsilon) = N(\xi_{\varepsilon+\eta}) \circ N(\zeta_\varepsilon) = \psi_{\varepsilon+\eta} \circ \varphi_\varepsilon$, which completes the claim.

C The instability of Mapper

In this section we briefly discuss how one may perceive that the simplicial complexes produced by Mapper may not admit a simple notion of stability.

As an example consider the situation in Figure 2. Consider for each $\delta > 0$ the domain X_δ shown in the figure (a topological graph with one loop), and the functions depicted in the figure: these are height functions f_δ and g_δ which differ by δ, that is $\|f_\delta - g_\delta\|_\infty = \delta$. The open cover is shown in the middle of the figure. Notice that the Mapper outputs for these two functions w.r.t. the same open cover of the co-domain $\mathbb{R}$ are different and that the situation can be replicated for each $\delta > 0$.

In contrast, one of the features of the Multiscale Mapper construction is that it is amenable to a certain type of stability under changes in the function and in the tower of covers. This situation is not surprising and is reminiscent of the pattern arising when comparing standard homology (and Betti numbers) computed on fixed simplicial complexes to persistent homology (and Betti barcodes/persistent diagrams) on tower of simplicial complexes (i.e. filtrations).

In what follows we will introduce a particular class of towers of covers that will be used to express some stability properties enjoyed by Multiscale Mapper.

D Good towers of covers

REMARK D.1. *We make the following remarks about the notion of (c, s)-good towers of covers:*

- *The underlying intuition is that an "ideal" tower of covers is one for which $c = 1$ and $s = 0$.*

- *A first example is given by the following construction: pick any $s \in [0, \text{diam}(Z)]$ and for $\varepsilon \geq s$ let*
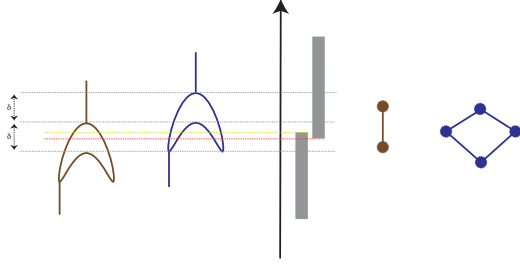
Figure 2: A situation in which Mapper yields two different answers for similar functions on the same domain. The construction can be carried out for each $\delta > 0$.

$\mathcal{U}_\varepsilon := \{B(z, \frac{\varepsilon}{2}), z \in Z\}$, *i.e. the collection of all ε-balls in Z. The maps $w_{\varepsilon,\varepsilon'}$ for $\varepsilon' \geq \varepsilon$ are defined in the obvious way: $B(z, \frac{\varepsilon}{2}) \mapsto B(z, \frac{\varepsilon'}{2})$, all $z \in Z$. Then, since any $O \subset Z$ is contained in some ball $B(o, \mathrm{diam}(O))$ for some $o \in O$, this means that $\{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ is $(2, s)$-good. Of course whenever Z is path connected and not a singleton, this tower of covers is infinite. A related finite construction is given next.*

- *A discrete set $P \subset Z$ is called an ν-sample of (Z, d_Z) if for any point $z \in Z$, $d_Z(z, P) \leq \nu$. Given a finite ν-sampling P of Z one can always find a $(3, 2\nu)$-good tower of covers of Z consisting, for each scale $\varepsilon \geq 2\nu$, of all balls $B(p, \frac{\varepsilon}{2})$, $p \in P$. Details are given in Appendix D.2. This family may not be space efficient. An example of how to construct a space efficient (c, s)-good tower of covers for a compact metric space (Z, d_Z) can be found in Appendix D.2.*

- *Conditions (1) and (2) mean that the tower begins at resolution s, and the resolution parameter ε controls the geometric characteristics of the elements of the cover. Parameter s is clearly related to the size/finiteness of $\mathfrak{U}$: if s is very small, then the number of sets in $\mathcal{U}_\varepsilon$ for ε larger than but close to s has to be large. Also, requiring s to be smaller than or equal to the diameter of Z ensures that condition 3 is not vacuously satisfied.*

- *Condition (3) controls the degree to which one can inject a given set in Z inside an element of the cover. The situation when $c > 1$ is consistent with having finite covers for all ε. On the other hand, $c = 1$ may require, for each ε, open covers with infinitely many elements.*

- *If $\mathfrak{U}$ is (c, s)-good, then it is (c', s')-good for all $c' \geq c$ and $s' \geq s$.*

A priori the conditions defining a (c, s)-good tower of covers do not guarantee that the sets O in Z with diameter smaller than the resolution s can be covered by some element W of some $\mathcal{W}_\varepsilon$, for some $\varepsilon \geq s$. The following proposition deals with this situation and is used later.

PROPOSITION D.1. *If Z is path connected and $\mathfrak{U}$ is a (c, s)-good tower of covers of Z, then whenever $O \subset Z$ is s.t. $\mathrm{diam}(O) < s$, there exists $W \in \mathcal{W}_{c \cdot (\mathrm{diam}(O)+2s)}$ s.t. $W \supseteq O$.*

REMARK D.2. *Note that for a small set O, $\mathrm{diam}(O) < s$, one obtains only that $O \subset W$ for some W in $\mathcal{W}_{c \cdot (\mathrm{diam}(O)+2s)}$. This is in contrast with what property 3 above guarantees for sets with diameter $\mathrm{diam}(O) \geq s$: the existence of $W \in \mathcal{W}_{c \cdot \mathrm{diam}(O)}$ with $O \subset W$.*

Proof. Indeed, since Z is path connected then one has that $\mathrm{diam}(O^s) \geq \min\{\mathrm{diam}(Z), s\} = s$ it follows by the definition that $\exists W' \in \mathcal{W}_{c \cdot \mathrm{diam}(O^s)}$ containing O. We conclude since $\mathrm{diam}(O^s) \leq \mathrm{diam}(O) + 2s$, $O \subset O^s$, and since there exists $W \in \mathcal{W}_{c(\mathrm{diam}(O)+2s)}$ containing W'.

We will henceforth assume that (Z, d_Z) is a compact path connected metric space.

D.1 The necessity of (c, s)-good tower of covers.
In Section §4.2, we aim to obtain a stability result for the multiscale mapper $\mathrm{MM}(\mathfrak{U}, f)$ with respect to perturbations of f. First, notice that we need to bound the resolution of an tower of covers from below by a parameter $s > 0$ to ensure its finiteness. We explain now why we need the second parameter c for the stability result.

Specifically, in what follows, we provide an example of a tower of covers of $\mathbb{R}$ which does not satisfy the condition 3 in (c, s)-goodness. This causes the persistence diagram $\mathrm{D}_1 \mathrm{MM}(\mathfrak{U}, f)$ to be **unstable** with respect to small perturbations of f.

Construction of a pathological tower of covers. Given $M > 0$ let $\mathbb{I} := [-M, M]$. For each $\varepsilon > 0$ and $k \in \mathbb{Z}$ let $I_k^{(\varepsilon)} := [k \cdot 2^{\lfloor \log_2^\varepsilon \rfloor}, (k+1) \cdot 2^{\lfloor \log_2^\varepsilon \rfloor}]$. Pick any $s \geq 0$ and consider the following tower $\mathfrak{W}$ of covers of $\mathbb{I}$ by closed intervals:

$$\mathfrak{W} = \left\{ \mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon,\varepsilon'}} \mathcal{W}_{\varepsilon'} \right\}_{s \leq \varepsilon \leq \varepsilon'};$$

where

$$\mathcal{W}_\varepsilon := \left\{ I_k^{(\varepsilon)} \mid k \in \mathbb{Z} \text{ s.t. } I_k^{(\varepsilon)} \cap \mathbb{I} \neq \emptyset \right\}.$$

The maps of covers $w_{\varepsilon,\varepsilon'} : \mathcal{W}_\varepsilon \to \mathcal{W}_{\varepsilon'}$ are defined below.

CLAIM 4. *For any $\varepsilon' > \varepsilon$, and each element $I_k^{(\varepsilon)} \in \mathcal{W}_\varepsilon$, there exists a unique $k' \in \mathbb{Z}$ such that $I_k^{(\varepsilon)} \subseteq I_{k'}^{(\varepsilon')}$.*

For each $k \in \mathbb{Z}$ we set $w_{\varepsilon,\varepsilon'}(I_k^{(\varepsilon)}) = I_{k'}^{(\varepsilon')}$ where $k' \in \mathbb{Z}$ is given the claim above.

Proof. (Proof of the claim) Write $q = \lfloor \log_2^\varepsilon \rfloor$ and $q' = \lfloor \log_2^{\varepsilon'} \rfloor = q + n$ for some $n \in \mathbb{N}$. We are going to produce $k' \in \mathbb{Z}$ such that $k' \cdot 2^{q'} \leq k \cdot 2^q$ and $(k+1) \cdot 2^q \leq (k'+1) \cdot 2^{q'}$, which will immediately imply that $I_k^{(\varepsilon)} \subseteq I_{k'}^{(\varepsilon')}$.

Choose k' to be the largest integer such that $k' \cdot 2^{q'} \leq k \cdot 2^q$. This means that $k' \in \mathbb{Z}$ is maximal amongst integers for which $k' \cdot 2^n \leq k$. By maximality, we have that $(k'+1) \cdot 2^n \geq k+1$. Then, this means that $(k'+1) \cdot 2^{q'} \geq (k+1) \cdot 2^q$, which implies that indeed $I_k^{(\varepsilon)} \subseteq I_{k'}^{(\varepsilon')}$.

In fact, $I_{k'}^{(\varepsilon')}$ is the unique interval in $\mathcal{W}_{\varepsilon'}$ that contains $I_k^{(\varepsilon)}$ due to the fact that intervals in $\mathcal{W}_{\varepsilon'}$ have disjoint interiors.

REMARK D.3. *By the argument in the proof of the above claim, it also follows that for any $s \leq \varepsilon < \varepsilon' < \varepsilon''$ and any $I \in \mathcal{W}_\varepsilon$, $w_{\varepsilon,\varepsilon''}(I) = w_{\varepsilon',\varepsilon''}(w_{\varepsilon,\varepsilon'}(I))$; that is, $w_{\varepsilon,\varepsilon''} = w_{\varepsilon',\varepsilon''} \circ w_{\varepsilon,\varepsilon'}$. Hence the maps $\{w_{\varepsilon,\varepsilon'} : \mathcal{W}_\varepsilon \to \mathcal{W}_{\varepsilon'}\}_{s \leq \varepsilon \leq \varepsilon'}$ are valid maps of covers, satisfying the conditions in Definition 3. Hence $\mathfrak{W}$ is a tower of (closed) covers.*

REMARK D.4. *This natural tower of covers $\mathfrak{W}$ is not (c, s)-good for any constant c. Specifically, consider an arbitrary small interval $(-r, r)$ for any $r > 0$. Clearly, there is no element in any $\mathcal{W}_\varepsilon$ that contains this interval. It turns out that the persistence diagram arising from multiscale mapper is unstable w.r.t. perturbations of the input function, as we will show by an example shortly. Note that, in contrast, stability of the persistence diagrams of multiscale mapper outputs is guaranteed for (c, s)-good tower of covers, as stated in Corollary 4.2 and Theorem 4.1.*

An example. We show how instability of multiscale mapper may arise for some choices of $\mathfrak{W}$ using the following example which is similar to the one presented in Figure 2. Let δ be any value larger than s, the smallest scale of the tower of covers $\mathfrak{W}$. Suppose we are given two functions defined on a graph G: $f, g : G \to \mathbb{R}$, as shown in Figure 3. It is clear that $\|f - g\|_\infty = \delta$.

$D_1 MM(\mathfrak{W}, f)$ consists of the point $(s, 2\delta)$, indicating that a non-null homologous loop exists at scale s
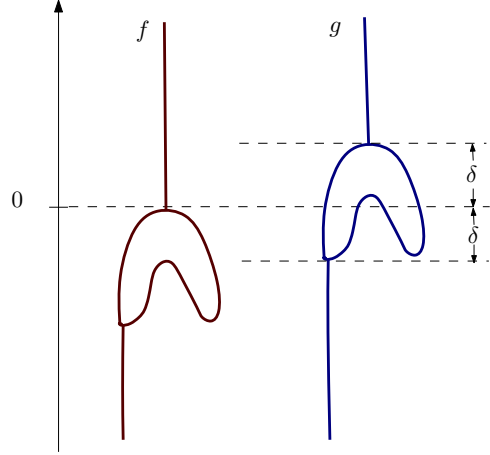

Figure 3: f and g.

in the pullback nerve complex $N(f^*(\mathcal{W}_s))$, but is killed at scale 2δ in the nerve complex $N(f^*(\mathcal{W}_{2\delta}))$ (specifically, the image of this loop under the simplicial map $f^*(w_{s,2\delta})$ becomes null-homologous).

However, for the function g, there is a loop created at the lowest scale s, but the homology class carried by this loop is never killed under the simplicial maps $g^*(w_{s,\varepsilon'}) : N(g^*(\mathcal{W}_s)) \to N(g^*(\mathcal{W}_{\varepsilon'}))$ for any $\varepsilon' > s$. Thus the persistence diagram $D_1 MM(\mathfrak{W}, g)$ consists of the point (s, ∞). Hence the two persistence diagrams $D_1 MM(\mathfrak{W}, f)$ and $D_1 MM(\mathfrak{W}, g)$ are not close under the bottleneck distance (in fact, the bottleneck distance between them is ∞), despite the fact that the functions f and g are δ-close , that is $\|f - g\|_\infty \leq \delta$.

REMARK D.5. *We remark that for clarity of presentation, in the above construction, each element in the covering $\mathcal{W}_\varepsilon$ is a closed interval. However, this example can be easily extended to open covers: Specifically, let ν be a sufficiently small positive value such that $\nu < s < \delta$. Then we can change each interval $I = [k \cdot 2^{\lfloor \log_2^\varepsilon \rfloor}, (k+1) \cdot 2^{\lfloor \log_2^\varepsilon \rfloor}] \in \mathcal{W}_\varepsilon$ to $I^\nu = (k \cdot 2^{\lfloor \log_2^\varepsilon \rfloor} - \nu, (k+1) \cdot 2^{\lfloor \log_2^\varepsilon \rfloor} + \nu)$. Note that for an arbitrary small interval $(-r, r)$ with $r > \nu$, there is no element in any $\mathcal{W}_\varepsilon$ that contains it, so that the resulting tower of covers is not (c, s)-good for any $c \geq 1$. Hence the example described in Figure 3 can be adapted whenever $\delta > \nu$; this leads to the following statement:*

PROPOSITION D.2. *Fix $s \geq 0$ and $W > s$. Then, for each $\frac{W}{2} > \delta > 0$ there exist (1) a topological graph G_δ, (2) two continuous functions $f_\delta, g_\delta : G_\delta \to [-W, W]$, and (3) a tower of covers $\mathfrak{W}$ of $[-W, W]$ satisfying the following properties:*

- $\|f_\delta - g_\delta\|_\infty = \delta$,

- $D_1MM(\mathfrak{W}, f_\delta) = \{(s, 2\delta)\}$ *and* $D_1MM(\mathfrak{W}, g_\delta) = \{(s, \infty)\}$.

D.2 Constructing a good towers of covers.

Suppose we are given a compact metric space (Z, d_Z) with bounded doubling dimension. We assume that we can obtain an ν-*sample* P of (Z, d_Z), which is a discrete set of points $P \subset Z$ such that for any point $z \in Z$, $d_Z(z, P) \leq \nu$. For example, if the input metric space Z is a d-dimensional cube in the Euclidean space $\mathbb{R}^d$, we can simply choose P to be the set of vertices from a d-dimensional lattice with edge length ν. For simplicity, assume that $\nu \leq 1$ (otherwise, we can rescale the metric to make this hold).

D.2.1 A simple (c, s)-good tower of covers.

Consider the following tower of covers $\mathfrak{W} = \{\mathcal{W}_\varepsilon \mid \varepsilon \geq 2\nu\}$ where $\mathcal{W}_\varepsilon := \{B_{\frac{\varepsilon}{2}}(u) \mid u \in P\}$. The associated maps of covers $w_{\varepsilon, \varepsilon'} : \mathcal{W}_\varepsilon \to \mathcal{W}_{\varepsilon'}$ simply sends each element $B_{\frac{\varepsilon}{2}}(u) \in \mathcal{W}_\varepsilon$ to the corresponding set $B_{\frac{\varepsilon'}{2}}(u) \in \mathcal{W}_{\varepsilon'}$. It is easy to see that $\mathfrak{W}$ is $(3, 2\nu)$-good. Indeed, given any $O \subseteq Z$ with diameter $R = \mathrm{diam}(O) \geq s = 2\nu$, pick an arbitrary point $o \in O$ and let $u \in P$ be a nearest neighbor of o in P. We then have that for any point $x \in O$,

$$d_Z(x, u) \leq d_Z(x, o) + d_Z(o, u) \leq R + \nu \leq \frac{3R}{2}.$$

That is, $O \subset B_{\frac{3R}{2}}(u) \in \mathcal{W}_{3R}$.

This tower of covers however has large size. In particular, as the scale ε becomes large, the number of elements in $\mathcal{W}_\varepsilon$ remains the same, while intuitively, a much smaller subset will be sufficient to cover Z. In what follows, we describe a different construction of a good tower of covers based on using the so-called *nets*.

D.2.2 A space-efficient (c, s)-good tower of covers.

Following the notations used by Har-Peled and Mendel in [21], we have the following:

PROPOSITION D.3. ([21]) *For any constant $\rho \geq 11$, and for any scale $\ell \in \mathbb{R}^+$, one can compute a ρ^ℓ-net $\mathcal{N}(\ell) \subseteq P$ in the sense that*

(i) *for any $p \in P$, $d_Z(p, \mathcal{N}(\ell)) \leq \rho^\ell$;*

(ii) *any two points $u, v \in \mathcal{N}(\ell)$, $d_Z(u, v) \geq \rho^{\ell-1}/16$, and*

(iii) *$\mathcal{N}(\ell') \subseteq \mathcal{N}(\ell)$ for any $\ell < \ell'$; that is, the net at a bigger scale is a subset of net at a smaller scale.*

Each $\mathcal{N}(\ell)$ is referred to as a net at scale ℓ.

From now on, we set $\varepsilon_i := 4(\rho + 1)^i$ for any positive integer $i \in \mathbb{Z}^+$. Consider the collection of nets $\{\mathcal{N}(i) \mid i \in \mathbb{Z}^+\}$ where $\mathcal{N}(i)$ is as described in Proposition D.3.

DEFINITION 13. *We define a tower of covers $\mathfrak{U} = \{U_{\varepsilon_i} \mid i \in \mathbb{Z}^+\}$ where:*
(D.1)
$$U_{\varepsilon_i} := \left\{B_{\frac{\varepsilon_i}{2}}(u) \mid u \in \mathcal{N}(i)\right\}, \text{ where } \varepsilon_i = 4(\rho + 1)^i.$$

The associated maps of covers $u_{\varepsilon_i, \varepsilon_j} : \mathcal{U}_{\varepsilon_i} \to \mathcal{U}_{\varepsilon_j}$, for any $i, j \in \mathbb{Z}^+$ with $i \neq j$, are defined as follows:

(1) *$u_{\varepsilon_i, \varepsilon_{i+1}} : \mathcal{U}_{\varepsilon_i} \to \mathcal{U}_{\varepsilon_{i+1}}$: For any element $U = B_{\frac{\varepsilon_i}{2}}(u) \in \mathcal{U}_{\varepsilon_i}$, we find the nearest neighbor $v \in \mathcal{N}(i+1)$ of u in $\mathcal{N}(i)$, and map U to $V = B_{\frac{\varepsilon_{i+1}}{2}}(v) \in \mathcal{U}_{\varepsilon_{i+1}}$.*

(2) *For $i < j - 1$, we set $u_{\varepsilon_i, \varepsilon_j}$ as the concatenation $u_{\varepsilon_{j-1}, \varepsilon_j} \circ u_{\varepsilon_{j-2}, \varepsilon_{j-1}} \circ \cdots \circ u_{\varepsilon_i, \varepsilon_{i+1}}$.*

REMARK D.6. *We note that the above tower of covers $\mathfrak{U}$ is discrete in the sense that we only consider covers $\mathcal{U}_\varepsilon$ for a discrete set of εs from $\{\varepsilon_i \mid i \in \mathbb{Z}^+\}$. However, one can easily extend it to a tower of covers $\mathfrak{U}^{\mathrm{ext}} = \{\mathcal{U}_\delta^{\mathrm{ext}}\}$ which is defined for all $\delta \in \mathbb{R}_+$ by declaring that $\mathcal{U}_\delta^{\mathrm{ext}} = \mathcal{U}_{\varepsilon_{\zeta(\delta)}}$ where $\zeta(\delta) = \max\{i \in \mathbb{Z} \mid \varepsilon_i \leq \delta\}$. One may define the cover maps in $\mathfrak{U}^{\mathrm{ext}}$ in a similar manner. For simplicity of exposition, in what follows we use a discrete tower of covers.*

CLAIM 5. *For any $i \in \mathbb{Z}^+$, U_{ε_i} forms a covering of the metric space Z (where P are sampled from).*

Proof. P is an ν-sampling of (Z, d_Z) with $\nu \leq 1$. Hence for any point $x \in Z$, there is a point $p_x \in P$ such that $d(x, p_x) \leq \varepsilon$. By Property (i) of Definition D.3, $d_Z(p_x, \mathcal{N}(i)) \leq \rho^i$. Hence $d_Z(x, \mathcal{N}(i)) \leq d_Z(x, p_x) + d_Z(p_x, \mathcal{N}(i)) \leq 1 + \rho^i \leq (\rho + 1)^i$. As such, there exists $u \in \mathcal{N}(i)$ such that $d_Z(x, u) \leq (\rho + 1)^i$; that is, $x \in U = B_{\frac{\varepsilon_i}{2}}(u) \in \mathcal{U}_{\varepsilon_i}$.

CLAIM 6. *For any $i < j$ and any $U \in \mathcal{U}_{\varepsilon_i}$, we have $U \subseteq u_{\varepsilon_i, \varepsilon_j}(U)$.*

Proof. We show that the claim holds true for $j = i + 1$, and the claim then follows from this and the construction of $u_{\varepsilon_i, \varepsilon_j}$ for $i < j - 1$.

Suppose $U = B_{\frac{\varepsilon_i}{2}}(u)$ for $u \in \mathcal{N}(i)$, and let $V = u_{\varepsilon_i, \varepsilon_{i+1}}(U)$. By construction, $V = B_{\frac{\varepsilon_{i+1}}{2}}(v)$ is such that v is the nearest neighbor of u in $\mathcal{N}(i+1)$. If $u = v$, then clearly $U \subseteq V$.

If $u \neq v$, by Property (i) of Definition D.3, $d_Z(u, v) \leq \rho^{i+1}$. At the same time, any point $x \in U$ is within $\frac{\varepsilon_i}{2} = 2(\rho + 1)^i$ distance to u. Hence

$$d_Z(x, v) \leq d_Z(x, u) + d_Z(u, v) \leq 2(\rho+1)^i + \rho^{i+1} < 2(\rho+1)^{i+1}.$$

It then follows that $U \subseteq V$.

THEOREM D.1. $\mathfrak{U}$ *as constructed above is a* (c, s)-*good tower of covers with* $c = s = 4(\rho + 1)$.

Proof. First, note that by Claim 5, each $\mathcal{U}_\varepsilon \in \mathfrak{U}$ is indeed a cover for (Z, d_Z). By Claim 6, the associated maps as constructed in Definition 13 are valid maps of covers. Hence $\mathfrak{U}$ is indeed a tower of covers for (Z, d_Z). We now show that $\mathfrak{U}$ is (c, s)-good.

First, by Eqn. D.1, each set U in $\mathcal{U}_{\varepsilon_i}$ obviously has diameter at most ε_i. Also, $\mathfrak{U}$ is s-truncated for $s = 4(\rho+1)$ since it starts with $\varepsilon_1 = s$. Hence properties 1 and 2 of Definition 7 hold. What remains is to show that property 3 of Definition 7 also holds.

Specifically, consider any $O \subset Z$ such that $\mathrm{diam}(O) \geq s$. Set $R = \mathrm{diam}(O)$ and let a be the unique integer such that

$$(\rho + 1)^{a-1} \leq R < (\rho + 1)^a.$$

Let $o \in O$ be any point in O, and $p \in P$ the nearest neighbor of o in P: By the sampling condition of P, we have $d_Z(o, p) \leq \nu < 1$. Let $u \in \mathcal{N}(a)$ be the nearest neighbor of p in the net $\mathcal{N}(a)$. By Definition D.3 (i), $d_Z(p, u) \leq \rho^a$. We thus have that, for any point $x \in O$,

$$\begin{aligned} d_Z(x, u) &\leq d_Z(x, o) + d_Z(o, p) + d_Z(p, u) \\ &\leq R + \varepsilon + \rho^a \\ &< (\rho + 1)^a + 1 + \rho^a \\ &\leq 2(\rho + 1)^a. \end{aligned}$$

In other words, $O \subseteq \mathrm{B}_{2(\rho+1)^a}(u) = \mathrm{B}_{\frac{\varepsilon_a}{2}}(u) \in \mathcal{U}_{\varepsilon_a}$.

On the other hand, since $R \geq (\rho + 1)^{a-1}$, we have that for $c = 4(\rho + 1)$,

$$\varepsilon_a = 4(\rho + 1)^a = c \cdot (\rho + 1)^{a-1} \leq cR = c \cdot \mathrm{diam}(O).$$

Space-efficiency of $\mathfrak{U}$. In comparison with the simple $(3, 2\rho)$-good tower of covers $\mathfrak{W}$ that we introduced at the beginning of this section, the main advantage of $\mathfrak{U}$ is its much more compact size. Intuitively, this comes from property (ii) of Proposition 13, which states that the points in the net $\mathcal{N}(i)$ are sparse and contain little redundancy. In fact, consider the covering $\mathcal{U}_{\varepsilon_i}$ at scale ε_i. It size (i.e, the cardinality of $\mathcal{U}_{\varepsilon_i}$) is close to optimal in the following sense:

For any ε, denote by $\mathcal{V}^*(\varepsilon)$ the smallest possible (in terms of cardinality) covering of (Z, d_Z) such that each element $V \in \mathcal{V}^*(\varepsilon)$ has diameter at most ε. Now, let $s^*(\varepsilon) := |\mathcal{V}^*(\varepsilon)|$ denote this optimal size for any covering of (Z, d_Z) by elements with diameter at most ε.

PROPOSITION D.4. *For any* $i \in \mathbb{Z}^+$, $s^*(\varepsilon_i) \leq |\mathcal{U}_{\varepsilon_i}| = |\mathcal{N}(i)| \leq s^*(\frac{\varepsilon_i}{16\rho})$.

Proof. The left inequality follows from the definition of $s^*(\varepsilon_i)$. We now prove the right inequality. Specifically, consider the smallest covering $\mathcal{V}^* = \mathcal{V}^*(\frac{\varepsilon_i}{16\rho})$ with $s^*(\frac{\varepsilon_i}{16\rho}) = |\mathcal{V}^*|$. By property (ii) of Proposition 13, each set $V \in \mathcal{V}^*$ can contain at most one point from $\mathcal{N}(i)$ since the diameter of V is at most $\frac{\varepsilon_i}{16\rho}$. At the same time, we know that the union of all sets in $\mathcal{V}^*$ will cover all points in $\mathcal{N}(i)$. The right inequality then follows.

E Proof of Corollary 4.2 and remarks

Proof. We use the notation of Proposition 4.3. Let $\mathfrak{U} = f^*(\mathfrak{W})$ and $\mathfrak{V} = g^*(\mathfrak{W})$. If $\max(1, s) = s$, then $\mathrm{R}_{\log}(\mathfrak{U})$ and $\mathrm{R}_{\log}(\mathfrak{V})$ are $\log(2c\max(s, \delta) + c)$-interleaved by Proposition 4.3 which gives a bound on the bottleneck distance of $\log(2c\max(s, \delta) + c)$ between the corresponding persistence diagrams [6]. In the case when $s < 1$, the bottleneck distance remains the same only for the 0-truncations of $\mathrm{R}_{\log}(\mathfrak{U})$ and $\mathrm{R}_{\log}(\mathfrak{V})$. By shifting the starting point of the two families to the left by at most $|\log s|$ can introduce barcodes of lengths at most $\log \frac{1}{s}$ or can stretch the existing barcodes to the left by at most $\log \frac{1}{s}$ for the respective persistence modules. To see this, consider the persistence module below where $\varepsilon_1 = \log s$:

$$\mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_1}))\big) \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_2}))\big) \to \cdots$$
$$\cdots \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_0))\big) \to \cdots \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_n}))\big)$$

A homology class born at any index in the range $[\log s, 0)$ either dies at or before the index 0 or is mapped to a homology class of $\mathrm{H}_k\big(N(f^*(\mathcal{U}_0))\big)$. In the first case we have a bar code of length at most $|\log s| = \log \frac{1}{s}$. In the second case, a bar code of the persistence module

$$\mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_1=0}))\big) \to \cdots \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_n}))\big)$$

starting at index $\log 1 = 0$ gets stretched to the left by at most $|\log s| = \log \frac{1}{s}$. The same conclusion can be drawn for the persistence module induced by $\mathrm{R}_{\log}(\mathfrak{V})$. Therefore the bottleneck distance between the respective persistence diagrams changes by at most $\log \frac{1}{s}$.

REMARK E.1. *The proposition and corollary above are in appearance somehow not satisfactory: imagine that $f = g$, then $\delta = 0$ but by invoking Proposition 4.3 instead of a 0-interleaving one obtains a $\big(\log(c(2s+1)) + \max(0, \log \frac{1}{s})\big)$-interleaving. Nevertheless, the claim of the proposition is almost tight for $\delta > 0$: in fact, by modifying the example in §D.1 for each $\delta > 0$ we can find a space X_δ, a (c, s)-good tower of covers $\mathfrak{U}$ of a closed interval $I \subset \mathbb{R}$, and a pair of functions $f_\delta, g_\delta : X \to I$ such that $\|f_\delta - g_\delta\|_\infty = \delta$ but such that the bottleneck*

distance between the persistence diagrams of $\mathrm{MM}(\mathfrak{U}, f_\delta)$ *and* $\mathrm{MM}(\mathfrak{U}, g_\delta)$ *is at least* $\max\{\log(cs), \log c, \log \frac{1}{s}\} = \Omega\big(\log(c(2s+1)) + \max(0, \log \frac{1}{s})\big).$

F   Missing details from Section 6

Construction of the maps ν_ε. Now we define the map $\nu_\varepsilon : \mathrm{M}_\varepsilon^{K_1} \to \mathrm{M}_\varepsilon$. Given a nerve complex M_ε (resp. $\mathrm{M}_\varepsilon^{K_1}$), we abuse the notation slightly and identify a vertex in M_ε with its corresponding connected component in $f^*(\mathcal{W}_\varepsilon)$ (resp. in $f_V^{*K_1}(\mathcal{W}_\varepsilon)$).

First, note that by construction of the combinatorial mapper, there is a natural map $u_W : \mathrm{cc}_{K_1}(f_V^{-1}(W)) \to \mathrm{cc}(f^{-1}(W))$ for any $W \in \mathcal{W}_\varepsilon$: For any $V \in \mathrm{cc}_{K_1}(f_V^{-1}(W))$, obviously, all vertices in V are K_1-connected in $f^{-1}(W)$. Hence there exists a unique set $U_V \in \mathrm{cc}(f^{-1}(W))$ such that $V \subseteq U_V$. We set $u_W(V) = U_V$ for each $V \in \mathrm{cc}_{K_1}(f_V^{-1}(W))$. The amalgamation of such maps for all $W \in \mathcal{W}_\varepsilon$ gives rise to the map $\nu_\varepsilon : f_V^{*K_1}(\mathcal{W}_\varepsilon) \to f^*(\mathcal{W}_\varepsilon)$, whose restriction to each $\mathrm{cc}_{K_1}(f_V^{-1}(W))$ is simply u_W as defined above. Abusing notation slightly, we use $\nu_\varepsilon : \mathrm{V}(\mathrm{M}_\varepsilon^{K_1}) \to \mathrm{V}(\mathrm{M}_\varepsilon)$ to denote the corresponding vertex map as well. Since for any set of vertices $V \in f_V^{*K_1}(\mathcal{W}_\varepsilon)$, we have that $V \subseteq \nu_\varepsilon(V) \subseteq |K|$. It then follows that non-empty intersections of sets in $f_V^{*K_1}(\mathcal{W}_\varepsilon)$ imply non-empty intersections of their images via ν_ε in $f^*(\mathcal{W}_\varepsilon)$. Hence this vertex map induces a simplicial map which we still denote by $\nu_\varepsilon : \mathrm{M}_\varepsilon^{K_1} \to \mathrm{M}_\varepsilon$.

Proof of Lemma 6.1. We have already proved in the main text that for all $\beta \in A_U$, $t_\varepsilon(V_\beta)$ has the same image $\hat{V}$. Using the same notation as in the main text, recall that

$$(*)\ \ V_\beta \subseteq U \text{ for any } \beta \in A_U, \quad \text{and} \quad \bigcup_{\beta \in A_U} V_\beta = \mathrm{V}(K) \cap U.$$

This implies that $\mathrm{V}(K) \cap U \subseteq \phi_\varepsilon(U)$. In fact, a similar argument can also be used to prove the following:

CLAIM 7. *The vertices of any simplex that intersects U will be contained in* $\phi_\varepsilon(U)$.

Now to prove Lemma 6.1, we need to show the following: given a k-simplex $\tau = \{p_0, p_1, \ldots, p_k\} \in \mathrm{M}_\varepsilon$, where each vertex p_j corresponds to set $U_j \subseteq |K|$, then we have that $\bigcap_{j=0}^k \phi_\varepsilon(U_j) \neq \emptyset$.

To prove this, take any point $x \in |K|$ such that $x \in \bigcap_{j=0}^k U_j$. Suppose x is contained in a simplex $\sigma \in K$. By Claim 7, the vertices of σ are contained in $\phi_\varepsilon(U_j)$, that is, $\mathrm{V}(\sigma) \subset \phi_\varepsilon(U_j)$, for any $j \in \{0, \ldots, k\}$. Hence $\bigcap_{j=0}^k \phi_\varepsilon(U_j) \supseteq \mathrm{V}(\sigma) \neq \emptyset$. The lemma then follows.