Lecture 7: Matrix Form for B-Spline Curves [1]

Quadric B-splines

We have seen earlier that some of the curve segments have same shape only shifted by some interval. For example, for $n = 5$ and $D = 3$, the basis functions $N_{2,3}, N_{3,3}$ had the same shape shifted by one interval. If we take $n = 7$ we will have $N_{2,3}, N_{3,3}, N_{4,3}, N_{5,3}$ same. In general, $N_{D-1,D}, ..., N_{n-D+1,D}$ will be same.

Uniform basis functions

Let us assume $D = 3$. Let us compute $N_{i,3}, N_{i+1,3}, N_{i+2,3}$ since each curve segment is determined by three basis functions for $D = 3$ as we have seen earlier. To compute these basis functions we will need $N_{i,1}, ..., N_{i+4,1}$ in the interval $i - 2 \leq u < i + 3$.

$$
\begin{aligned}
N_{i,1}(u) &= 1 \ if \ i-2 \leq u < i-1 \\
&= 0 \ otherwise \\
N_{i+1,1}(u) &= 1 \ if \ i-1 \leq u < i \\
&= 0 \ otherwise \\
&. \\
&. \\
N_{i+4,1}(u) &= 1 \ if \ i+2 \leq u \leq i+3 \\
&= 0 \ otherwise
\end{aligned}
$$

This gives:

$$
\begin{aligned}
N_{i,2}(u) &= (u-i+2)N_{i,1}(u) + (i-u)N_{i+1,1}(u) \\
N_{i+1,2}(u) &= (u-i+1)N_{i+1,1}(u) + (i+1-u)N_{i+2,1}(u) \\
N_{i+2,2}(u) &= (u-i)N_{i+2,1}(u) + (i+2-u)N_{i+3,1}(u) \\
N_{i+3,2}(u) &= (u-i-1)N_{i+3,1}(u) + (i+3-u)N_{i+4,1}(u)
\end{aligned}
$$

We obtain $N_{i,3}$ as:

$$
\begin{aligned}
N_{i,3}(u) &= \frac{1}{2}[(u-i+2)^2 N_{i,1}(u) + (u-i+2)(i-u)N_{i+1,1}(u) \\
&\quad +(i+1-u)(u-i+1)N_{i+1,1}(u) + (i+1-u)^2 N_{i+2,1}(u)] \\
N_{i+1,3}(u) &= \frac{1}{2}[(u-i+1)^2 N_{i+1,1}(u) + (u-i+1)(i+1-u)N_{i+2,1}(u) \\
&\quad +(i+2-u)(u-i)N_{i+2,1}(u) + (i+2-u)^2 N_{i+3,1}(u)] \\
N_{i+2,3}(u) &= \frac{1}{2}[(u-i)^2 N_{i+2,1}(u) + (u-i)(i+2-u)N_{i+3,1}(u) \\
&\quad +(i+3-u)(u-i-1)N_{i+3,1}(u) + (i+3-u)^2 N_{i+4,1}(u)]
\end{aligned}
$$

[1] Note by Tamal K. Dey

Now we have the equation of the curve segment for the interval $i \leq u < i+1$.

$$\begin{aligned} \mathbf{p}(u) &= \frac{1}{2}(i+1-u)^2\mathbf{p}_i \\ &+\frac{1}{2}[(u-i+1)(i+1-u)+(i+2-u)(u-i)]\mathbf{p}_{i+1} \\ &+\frac{1}{2}(u-i)^2\mathbf{p}_{i+2} \end{aligned}$$

If we reparameterize the curve for $0 \leq u < 1$, we get:

$$\mathbf{p}(u) = \frac{1}{2}[(1-u)^2\mathbf{p}_i + (-2u^2+2u+1)\mathbf{p}_{i+1} + u^2\mathbf{p}_{i+2}]$$

Matrix Form

We now write the curve segment between $i-1 \leq u < i$ as $\mathbf{p}_i(u)$ and get:

$$\mathbf{p}_i(u) = \frac{1}{2}[u^2 \ u \ 1] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix} \quad for \ i \in [1, n-1]$$

So, a second degree uniform B-spline curve is given by $\mathbf{p}_i(u) = \mathbf{U}\mathbf{M_s}\mathbf{P}$ for $i \in [1, n-1]$ where

$$\mathbf{M_s} = \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\mathbf{P} = [\mathbf{p}_{i-1} \ \mathbf{p}_i \ \mathbf{p}_{i+1}]^T$.