Progress in Geometric Transversal Theory

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Abstract

Let $\mathcal{A}$ be a family of convex sets in $\mathbb{R}^d$. A line transversal to $\mathcal{A}$ is a line which intersects every member of $\mathcal{A}$. More generally, a k-tranversal to $\mathcal{A}$ is an affine subspace of dimension k which intersects every member of $\mathcal{A}$. This paper discusses recent progress in geometric transversal theory, including necessary and sufficient conditions, Helly-type theorems, piercing or Gallai numbers, topological and combinatorial structure, and induced orderings.

1 Introduction

A k-transversal to a family of convex sets in $\mathbb{R}^d$ is an affine subspace of dimension k, called a *k-flat*, (such as a point, line, plane or hyperplane) which intersects every member of the family. The study of k-transversals dates back to papers by Vincensini in 1935 [63] and Santaló in 1940 [59] on line transversals in the plane but really developed only in the late 1950's and early 1960's with contributions by Danzer, Debrunner, Grünbaum, Hadwiger, Klee and Valentine [14]. There was incremental progress since then, most notably Eckhoff's thesis in 1969 [18] and work by Katchalski and Lewis in the early eighties [40, 42, 43, 50]. In the last ten years there has been an explosion of results, including necessary and sufficient conditions for hyperplane transversals, Helly-type theorems for line transversals to translates, piercing or Gallai numbers for hyperplane transversals, bounds on the complexity of the space of transversals and the number of orders induced by transversals, and algorithms for finding transversals. This paper will cover some of the major recent results in geometric transversal theory and describe some of the open problems suggested by these results.

2 The Space of k-Transversals

Let $\mathcal{A}$ be a family of convex sets in $\mathbb{R}^d$. The set of k-transversals to $\mathcal{A}$ forms a topological space, denoted $\mathcal{T}_k^d(\mathcal{A})$, lying in the "affine Grassmannian" of all

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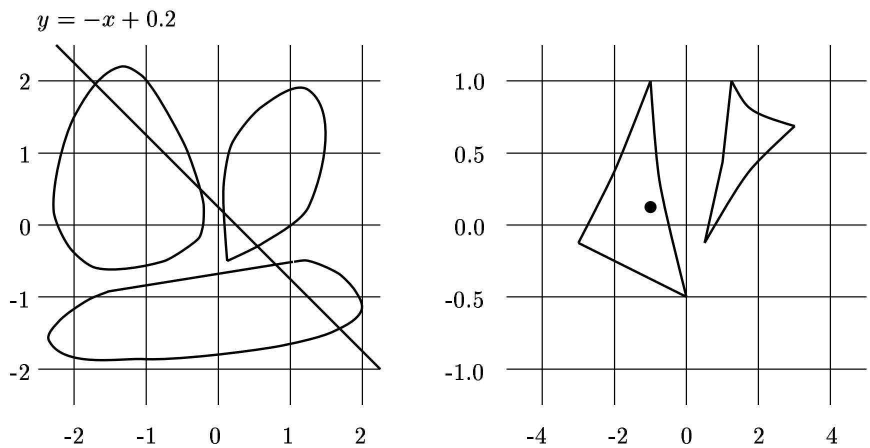

Figure 1: A family $\mathcal{A}$ of convex sets and its transversal space $\mathcal{T}_1^2(\mathcal{A})$.

k-flats in $\mathbb{R}^d$. This space can be easily visualized for line transversals in the plane by mapping each non-vertical line transversal $\{(x, y) : y = mx + b\}$ to the point (m, b) in dual space. (See Figure 1.)

As the example in Figure 1 shows, the space of transversals is not necessarily connected, even for line transversals in the plane. The space of line transversals to a family of n convex sets in the plane can have as many as $2n - 2$ connected components [24, 47].

Let $\mathcal{A}$ be a family of pairwise disjoint convex sets in $\mathbb{R}^d$. A directed line transversal intersects the elements of $\mathcal{A}$ in a fixed order. An undirected line induces a pair of linear orderings on $\mathcal{A}$ corresponding to its two possible orientations. Katchalski [41] studied pairs of linear orderings induced by line transversals and called them *geometric permutations.*

If two line transversals induce different geometric permutations on $\mathcal{A}$, then they must lie in different connected components in the space of line transversals to $\mathcal{A}$. Thus each connected component is associated with a single geometric permutation of $\mathcal{A}$. In the plane, the converse is also true. Two lines which induce the same geometric permutation on a finite family $\mathcal{A}$ of pairwise disjoint compact convex sets in $\mathbb{R}^2$ must lie in the same connected component in $\mathcal{T}_1^2(\mathcal{A})$ [66]. In $\mathbb{R}^3$, this is no longer true. Two lines which induce the same geometric permutation on a family $\mathcal{A}$ in $\mathbb{R}^3$ may lie in different connected components in $\mathcal{T}_1^3(\mathcal{A})$.

The generalization of geometric permutations to k-transversals requires a generalization of linear orderings to higher dimensions. In the early eighties, Goodman and Pollack [26] introduced and explored the notion of the *order type* of a set of points which is the family of orientations (-1, 0 or $+1$) of its $(d+1)$-tuples. For example, the order type of a set of points in the plane is the family of left or right turns (or no turn) made by every three points of the set. The order type of a set of points in $\mathbb{R}^d$ describes the relative positions of the points in $\mathbb{R}^d$ in the same way that a linear ordering of points in $\mathbb{R}$ captures their relative positions in $\mathbb{R}$.

Order types are special cases of *acyclic oriented matroids.* A rank $d+1$ acyclic oriented matroid on a set $\mathcal{A}$ is a family of orientations of the $(d+1)$-tuples of $\mathcal{A}$ with the property that the orientations of the $(d+1)$-tuples of every subset of $\mathcal{A}$ of size $d+3$ match the orientations of the $(d+1)$-tuples of some set of $d+3$ points in $\mathbb{R}^d$ [11, p.140]. In other words, an acyclic oriented matroid behaves locally (i.e., for subsets of size $d+3$) like an order type of points in $\mathbb{R}^d$. If all the $(d+1)$-tuples have non-zero orientation, this property suffices to define an oriented matroid; otherwise, additional conditions relating the zero and non-zero orientations are needed. (A rank $d+1$ oriented matroid which is not acyclic "locally" matches the orientation of vectors in $\mathbb{R}^{d+1}$.) Not every acyclic oriented matroid can be represented as the order type of some set of points. Those that can are called *realizable.* (See [11] for an introduction to oriented matroid theory.)

Let τ be an oriented k-transversal of a family $\mathcal{A}$ of convex sets. For each $a \in \mathcal{A}$, choose a point $p_a \in a \cap \tau$. Define a rank $k+1$ realizable acyclic oriented matroid on $\mathcal{A}$ by setting the orientation of every $(k+1)$-tuple of $\mathcal{A}$ to equal the orientation in τ of the corresponding $(k+1)$-tuple of points of $\{p_a\}$. Goodman and Pollack noted that if no $k+1$ members of $\mathcal{A}$ have a $(k-1)$-transversal, then none of the orientations are zero and so this oriented matroid is independent of the choices of the points p_a. Thus if no $k+1$ members of $\mathcal{A}$ have a $(k-1)$-transversal, then τ induces an oriented matroid on $\mathcal{A}$.

A family $\mathcal{A}$ of convex sets is *k-separated* if no $k+2$ members of $\mathcal{A}$ have a k-transversal. Note that a family of convex sets is 0-separated if and only if the sets are pairwise disjoint. An oriented k-transversal induces a unique rank $k+1$ oriented matroid on a $(k-1)$-separated family of convex sets.

Just as a line induces a pair of linear orderings on a family of pairwise disjoint sets, a k-transversals induces a pair of oriented matroids on a $(k-1)$-separated family corresponding to its two orientations. Unfortunately, the term geometric permutation is misleading when applied to pairs of oriented matroids which do not resemble permutations in any way.

Two k-transversals which induce different pairs of oriented matroids on $\mathcal{A}$ must lie in different connected components of $\mathcal{T}_k^d(\mathcal{A})$. The converse is not true in general but does hold for hyperplane transversals [66]. Just as line transversals in $\mathbb{R}^2$ which induce the same geometric permutation lie in the same connected component, two $(d-1)$-transversals which induce the same pair of oriented matroids on a finite $(d-2)$-separated family $\mathcal{A}$ of compact convex sets in $\mathbb{R}^d$ lie in the same connected component of $\mathcal{T}_{d-1}^d(\mathcal{A})$.

3 Necessary and Sufficient Conditions

A 0-transversal to a family $\mathcal{A}$ of convex sets is simply a point common to all the sets. The well-known Helly's Theorem can be reformulated as necessary and sufficient conditions for the existence of point transversals.

Theorem 1 Helly's Theorem [36]. *Let $\mathcal{A}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$. Family $\mathcal{A}$ has a point transversal (a point in common) if and only if every $d+1$ members of $\mathcal{A}$ have a point transversal (a point in common.)*

Helly's Theorem motivated many other theorems of a similar form: A family of objects has property $\mathcal{P}$ if and only if every m members of the family have property $\mathcal{P}$. These theorems are sometimes called *Helly-type* theorems.

Vincensini's 1935 paper [63] gave a Helly-type theorem for the existence of line transversals in the plane, a theorem which unfortunately was false. In 1940 Santaló [59] showed that for any m there is a family $\mathcal{A}$ of convex sets in $\mathbb{R}^2$ such that every m members of $\mathcal{A}$ have a line transversal but $\mathcal{A}$ has no line transversal. Necessary and sufficient conditions were given by Santaló and others for special families such as parallelotopes with edges parallel to the coordinate axes [59], but it was not until 1957 that Hadwiger gave the first general conditions for the existence of line transversals in the plane. Hadwiger's Transversal Theorem is:

Theorem 2 Hadwiger's Transversal Theorem [33]. *Let $\mathcal{A}$ be a finite family of pairwise disjoint convex sets in $\mathbb{R}^2$. Family $\mathcal{A}$ has a line transversal if and only if there exists a linear ordering of $\mathcal{A}$ such that every three members of $\mathcal{A}$ are intersected by a directed line in the given order.*

As was noted in Section 2, the space of line transversals to a family $\mathcal{A}$ of convex sets in the plane can have many different connected components. To some extent, this explains why there is no Helly-type theorem for line transversals in the plane. The transversal space is simply too complicated. However, every connected component in the space of line transversals of $\mathcal{A}$ is associated with a unique pair of orderings of $\mathcal{A}$. By prespecifying a linear ordering, Theorem 2 focuses on the existence of a single connected component. Interestingly enough, Theorem 2 does not guarantee that some directed line transversal intersects $\mathcal{A}$ in the given linear ordering. Every four members of $\mathcal{A}$ must be intersected by a directed line in the given order to ensure that result.

Let $\mathcal{A}$ be a $(d-1)$-separated family of compact convex sets in $\mathbb{R}^d$. Every connected component in the space of hyperplane transversals of $\mathcal{A}$ is associated with a unique pair of oriented matroids on $\mathcal{A}$. By prespecifying an oriented matroid, one can focus on the existence of a single connected component of $\mathcal{T}^d_{d-1}(\mathcal{A})$. Using order types (realizable acyclic oriented matroids) Goodman and Pollack in 1988 generalized Hadwiger's Transversal Theorem to hyperplane transversals. Their proof only uses local properties of order types and so applies equally well to acyclic oriented matroids as is stated here.

Theorem 3 [27]. *Let $\mathcal{A}$ be a finite $(d-2)$-separated family of convex sets in $\mathbb{R}^d$. Family $\mathcal{A}$ has a hyperplane transversal if and only if there exists a rank $d+1$ acyclic oriented matroid of $\mathcal{A}$ such that every $d+1$ members of $\mathcal{A}$ are intersected by an oriented hyperplane consistently with that oriented matroid.*

An oriented hyperplane h intersects a family $\mathcal{B} \subseteq \mathcal{A}$ consistently with an oriented matroid $\mathcal{M}$ on $\mathcal{A}$ if the oriented matroid $\mathcal{M}'$ induced by h on $\mathcal{B}$ is a submatroid of $\mathcal{M}$, i.e., the orientation of every $(d+1)$-tuple in $\mathcal{M}'$ equals the orientation of the corresponding $(d+1)$-tuple in $\mathcal{M}$. Theorem 3 does not guarantee that $\mathcal{A}$ has an oriented transversal which induces the given oriented matroid on $\mathcal{A}$.

The condition that $\mathcal{A}$ is $(d-2)$-separated generalizes the pairwise disjointness condition in Hadwiger's original theorem. In 1989, Wenger [65] removed this condition for line transverals in $\mathbb{R}^2$ and subsequently Pollack and Wenger [57] removed the condition for hyperplane transversals in $\mathbb{R}^d$. Their theorem also generalizes conditions by Katchalski [40] for the existence of hyperplane transversals based on the existence of line transversals to every three sets and applies to families of connected sets, not just convex ones.

Theorem 4 [57]. *Let $\mathcal{A}$ be a finite family of connected sets in $\mathbb{R}^d$. Family $\mathcal{A}$ has a hyperplane transversal if and only if for some k, $0 \leq k < d$, there exists a rank $k+1$ realizable acyclic oriented matroid of $\mathcal{A}$ such that every $k+2$ members of $\mathcal{A}$ are intersected by an oriented k-flat consistently with that oriented matroid.*

Since $\mathcal{A}$ may not be $(k-1)$-separated, an oriented k-flat τ which intersects $\mathcal{B} \subseteq \mathcal{A}$ may not induce a unique oriented matroid on $\mathcal{B}$. Nevertheless, we say that a k-flat τ intersects $\mathcal{B} \subseteq \mathcal{A}$ *consistently* with an oriented matroid $\mathcal{M}$ on $\mathcal{A}$ if for every $b \in \mathcal{B}$ there is a point $p_b \in b \cap \tau$ such that the orientations of $(k+1)$-tuples in $\{p_b\}$ match the orientations of corresponding $(k+1)$-tuple of $\mathcal{M}$.

The proof by Hadwiger of Theorem 2 and the generalization by Goodman and Pollack entails shrinking the convex sets in $\mathcal{A}$ until some $d+1$ sets have only a single hyperplane transversal h consistent with the given oriented matroid. Those $d+1$ sets pin the hyperplane h and one can show that h intersects every other convex set in $\mathcal{A}$.

The proof by Pollack and Wenger of Theorem 4 is quite different. Each rank $k+1$ realizable acyclic oriented matroid corresponds to the family of orientations of some set P of points in $\mathbb{R}^k$. Pollack and Wenger construct an antipodal mapping from the set of unit vectors in $\mathbb{R}^d$ to $\mathbb{R}^k$ based on the family of convex sets $\mathcal{A}$ and the point set P. By the Borsuk-Ulam theorem, one of these vectors maps to zero. The proof concludes by showing that this vector is the normal of a hyperplane transversal of $\mathcal{A}$.

The proof by Pollack and Wenger involves explicitly representing the oriented matroid by a set of points $P \subseteq \mathbb{R}^k$. Thus the oriented matroids in Theorem 4 must be realizable. Anderson and Wenger removed this realizability condition to give the theorem in its most general form:

Theorem 5 [9]. *Let $\mathcal{A}$ be a finite family of connected sets in $\mathbb{R}^d$. Family $\mathcal{A}$ has a hyperplane transversal if and only if for some k, $0 \leq k < d$, there exists a rank $k+1$ acyclic oriented matroid of $\mathcal{A}$ such that every $k+2$ members of $\mathcal{A}$ are intersected by an oriented k-flat consistently with that oriented matroid.*

A rank $k+1$ oriented matroid can be represented by an arrangement of oriented pseudospheres (topological spheres whose intersections are topological spheres of the proper dimension) in $\mathbb{S}^k$, the k-sphere [11, Chapter 5]. Based on the family of convex sets $\mathcal{A}$, Anderson and Wenger construct a lower semi-continuous antipodal mapping from the set of unit vectors $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ to faces in this pseudosphere representation in $\mathbb{S}^k$. They prove that some unit vector must map to the null face, $\emptyset$, and that this vector is the normal of a hyperplane transversal of $\mathcal{A}$.

Hadwiger's Transversal Theorem and its generalizations give conditions for the existence of hyperplane transversals ((d–1)-transversals) in $\mathbb{R}^d$. What about k-transversals for values of k other than $d-1$? Two lines may induce the same pair of orderings on a family $\mathcal{A}$ in $\mathbb{R}^3$ but lie in different connected components in $\mathcal{T}_1^3(\mathcal{A})$. Thus specifying a given ordering may not isolate a single connected component in $\mathcal{T}_1^3(\mathcal{A})$. Not suprisingly, Hadwiger's Transversal Theorem does not generalize to line transversals in $\mathbb{R}^3$. Aronov, Goodman, Pollack and Wenger showed that for any number m there exists a family $\mathcal{A}$ of convex sets in $\mathbb{R}^3$ and a linear ordering on $\mathcal{A}$ such that every m convex sets are met by a line consistently with that linear ordering but $\mathcal{A}$ has no line transversal [29].

Problem 1. *Give necessary and sufficient conditions for the existence of line transversals to a family of convex sets in $\mathbb{R}^3$ (or more generally k-transversals to a family of convex sets in $\mathbb{R}^d$ where $0 < k < d-1$.)*

4 Helly-Type Transversal Theorems

Santaló [59] showed that there is no Helly-type theorem for line transversals to families of convex sets in the plane. Hadwiger and Debrunner [35] proved that there were no such theorems even if the convex sets were pairwise disjoint and Lewis [50] constructed counter-examples for families of pairwise disjoint line segments.

However, for special types of families there are Helly-type theorems for line transversals. In a 1957 paper, Danzer [13] proved that if $\mathcal{A}$ was a family of pairwise disjoint congruent circles and every five members of $\mathcal{A}$ have a line transversal then $\mathcal{A}$ has a line transversal. A year later Grünbaum [32] gave a similar theorem for pairwise disjoint congruent squares and conjectured that the theorem was true for any family of pairwise disjoint translates of a convex set.

Starting around 1980, Katchalski worked on Grünbaum's conjecture, publishing a number of papers on line transversals [40, 42, 43]. His efforts resulted in a 1986 paper [41] in which he proved a weak form of Grünbaum's conjecture showing that if every 128 members of a family of pairwise disjoint translates had a line transversal then the family had a line transversal. Finally, in 1988, thirty years after Grünbaum published his conjecture, Tverberg presented a proof.

Theorem 6 [62]**.** *Let $\mathcal{A}$ be a finite family of pairwise disjoint translates of a convex set in $\mathbb{R}^2$. Family $\mathcal{A}$ has a line transversal if and only if every five members of $\mathcal{A}$ have a line transversal.*

Tverberg first proved the conjecture for families of six pairwise disjoint translates. These six translates can be characterized by six points in the plane representing the six directions of translation. A careful analysis of the different orders in which lines intersect five of the six translates and of different configurations of six points in the plane proves the conjecture for families of size six. Larger families of more than six convex sets can be transformed by shrinking into families where some subset of five translates has a unique line transversal. Since Grünbaum's conjecture is established for six translates, this unique line transversal must meet every other translate and thus must be a line transversal for the entire family.

Katchalski's proof of a weak form of Theorem 6 is quite different. A family of n pairwise disjoint convex sets in $\mathbb{R}^2$ can have up to $2n-2$ different geometric permutations (Theorem 20.) However, Katchalski, Lewis and Liu [44] proved that a family $\mathcal{A}$ of pairwise disjoint translates can have at most eight geometric permutations, independent of the size of the family. They later reduced the constant eight to three [45, 46]. In addition, any two of these geometric permutations must agree on all but five of their elements. These conditions on geometric permutations can be used to construct an ordering of $\mathcal{A}$ from the line transversals of subsets of $\mathcal{A}$ of size 128 such that every three members of $\mathcal{A}$ are intersected by a directed line in the given order. Applying Hadwiger's Transversal Theorem, Theorem 2, gives the desired result.

As discussed in Section 2, each geometric permutation of $\mathcal{A}$ corresponds to a single connected component in $\mathcal{T}_1^2(\mathcal{A})$. Thus, Katchalski, Lewis and Liu implicitly proved that $\mathcal{T}_1^2(\mathcal{A})$ has at most three connected components if $\mathcal{A}$ is a family of pairwise disjoint translates. This partly explains why there is a Helly-type theorem for line transversals to pairwise disjoint translates.

A convex set is *ρ-stubby*, $\rho \geq 1$, if it is contained in a ball of radius ρ and contains a ball of radius one. Katchalski's proof of his Helly-type theorem generalizes to finite families $\mathcal{A}$ of pairwise disjoint ρ-stubby sets. For each ρ there is a constant c_ρ such that a finite family $\mathcal{A}$ of pairwise disjoint ρ-stubby convex sets has a line transversal if and only if every c_ρ members of $\mathcal{A}$ have a line transversal. Any family of pairwise disjoint translates can be mapped by affine transformation to a 2-stubby family [38]. Thus the Helly-type theorem for line transversals of ρ-stubby sets leads directly to a Helly-type theorem for line transversals of translates.

Recently, Robert [58] replaced the pairwise disjointness condition in Theorem 6 by a condition that the intersection of any j translates is empty. For every j there is a constant c_j such that if $\mathcal{A}$ is a finite family of convex translates and the intersection of every j translates in $\mathcal{A}$ is empty, then $\mathcal{A}$ has a line transversal if and only if every c_j translates have a line transversal.

Under certain conditions, Helly-type theorems can be used to derive linear expected time algorithms for related algorithmic problems. In [61], Sharir and Welzl described a technique called Generalized Linear Programming which they applied to many problems related to convex programming. In particular, they gave an algorithm for finding a point in the intersection of n half-spaces in $\mathbb{R}^d$ and proved that their algorithm ran in $O(d^3 2^d n)$ expected time. This bound is linear in the number of half-spaces, n, although exponential in d. (Matoušek, Sharir and Welzl [53] subsequently proved a subexponential $O(nde^{4\sqrt{d \ln(n+1)}})$

bound on the expected running time of this algorithm.) Amenta [7] showed that Generalized Linear Programming could be applied to solve in linear expected time many other algorithmic problems related to Helly-type theorems including the problem of finding a line transversal for translates. (See also [15, 16, 17, 25, 54, 55] for deterministic algorithms which solve specific Helly-type problems in linear time.)

Katchalski conjectured that Theorem 6 generalizes to families $\mathcal{A}$ of translates in higher dimensions. If this were true, the number of connected components of $\mathcal{T}_k^d(\mathcal{A})$ and the number of oriented matroids on $\mathcal{A}$ induced by oriented k-transversals is probably bounded by a fixed constant depending only on k and d.

Problem 2. *Do there exist numbers m, m' and m'' such that for any family $\mathcal{A}$ of pairwise disjoint translates of a convex set in $\mathbb{R}^3$:*

- *$\mathcal{A}$ has a line transversal if and only if every m members of $\mathcal{A}$ have a line transversal;*
- *the number of geometric permutations of $\mathcal{A}$ is at most m';*
- *the number of connected components of $\mathcal{T}_1^3(\mathcal{A})$ is at most m''?*

(More generally, do there exist similar bounds for k-transversals to $(k-1)$-separated families of translates in $\mathbb{R}^d$?)

5 Piercing Numbers

In 1957, Hadwiger and Debrunner gave the following variation of Helly's Theorem:

Theorem 7 [34]**.** *For every $p \geq q \geq d+1$ where $p(d-1) < (q-1)d$: If $\mathcal{A}$ is a finite family of at least p convex sets in $\mathbb{R}^d$ and out of every p members of $\mathcal{A}$ some q have a point in common, then some set of $p-q+1$ points intersects every member of $\mathcal{A}$.*

The value of $p-q+1$ is tight and cannot be reduced. The smallest number of points required to intersect every member of $\mathcal{A}$ is called the *piercing number* or sometimes the *Gallai number* of $\mathcal{A}$ after a question of T. Gallai on the smallest number of needles required to pierce all members of any family of pairwise intersecting circular disks in $\mathbb{R}^2$.

Without the condition $p(d-1) < (q-1)d$ it is not necessarily true (and sometimes false) that some set of $p-q+1$ points intersects every member of $\mathcal{A}$ but Hadwiger and Debrunner conjectured that for every p and q one could replace $p-q+1$ by some constant c for which the theorem would hold. In the following ten years, this was shown for families of axes parallel parallelotopes [35] and for families of homothets [64] but it was not until 1992 that Alon and Kleitman presented a proof for the general case.

Theorem 8 [6]. *For every $p \geq q \geq d+1$, there exists a positive integer $c_{p,q,d}$ such that: If $\mathcal{A}$ is a finite family of at least p convex sets in $\mathbb{R}^d$ and out of every p members of $\mathcal{A}$ some q have a point in common, then some set of $c_{p,q,d}$ points intersects every member of $\mathcal{A}$.*

Let $\mathcal{A}$ be a family of $n \geq p$ convex sets in $\mathbb{R}^d$ such that out of every p members of $\mathcal{A}$ some q have a point in common. Alon and Kleitman use Katchalski and Liu's fractional version of Helly's Theorem [48] to show that some point intersects at least βn members of $\mathcal{A}$ where β is a fixed constant depending solely on p, q and d. (See also [4, 20, 39] for sharp quantitative proofs.) They then apply duality in linear programming to construct a family Y of points such that every member of $\mathcal{A}$ intersects at least $\beta|Y|$ elements of Y. Finally, using weak ϵ-nets for convex hulls they find a set X whose size is a fixed constant depending solely on β and d such that the convex hull of any subset of Y of size $\beta|Y|$ contains some member of X. Since every convex set in $\mathcal{A}$ contains at least $\beta|Y|$ elements of Y, every convex set in $\mathcal{A}$ intersects X. (See [49] for a new, purely combinatorial proof by the same authors.)

Although Helly's theorem does not generalize to hyperplane transversals, Alon and Kalai showed that Theorem 8 does.

Theorem 9 [5]. *For every $p \geq q \geq d+1$, there exists a positive integer $c_{p,q,d}$ such that: If $\mathcal{A}$ is a finite family of at least p convex sets in $\mathbb{R}^d$ and out of every p members of $\mathcal{A}$ some q have a hyperplane transversal, then some set of $c_{p,q,d}$ hyperplanes intersects every member of $\mathcal{A}$.*

The proof of Theorem 9 follows the proof of Theorem 8. Let $\mathcal{A}$ be a family of $n \geq p$ convex sets in $\mathbb{R}^d$ such that out of every p members of $\mathcal{A}$ some q have a hyperplane transversal. Alon and Kalai show that some hyperplane intersects at least βn members of $\mathcal{A}$ and for some family Y of hyperplanes every member of $\mathcal{A}$ intersects at least $\beta|Y|$ elements of Y. They construct a β-cutting for Y, i.e., a set of d-dimensional simplices partitioning $\mathbb{R}^d$ whose size is a fixed constant depending solely on β and d such that the interior of every simplex intersects fewer than $\beta|Y|$ members of Y. (See [52].) Every member of $\mathcal{A}$ intersects at least $\beta|Y|$ members of Y so every member of $\mathcal{A}$ intersects at least one simplex facet. If X is the set of hyperplanes containing the simplex facets, then every member of $\mathcal{A}$ intersects X.

Eckhoff gave almost exact minimal values for $c_{p,p,2}$.

Theorem 10 [19]. *Let $\mathcal{A}$ be a finite family of convex sets in $\mathbb{R}^2$. If every four members of $\mathcal{A}$ have a line transversal, then there is a set of two lines which intersects every member of $\mathcal{A}$.*

Theorem 11 [21]. *Let $\mathcal{A}$ be a finite family of convex sets in $\mathbb{R}^2$. If every three members of $\mathcal{A}$ have a line transversal, then there is a set of four lines which intersects every member of $\mathcal{A}$.*

Eckhoff conjectured, although he could not prove, that the number four in the conclusion of Theorem 11 could be reduced to three.

Alon and Kalai conjectured that Theorem 9 could be generalized to k-transversals in $\mathbb{R}^d$. A first step would be to show that the existence of a k-transversal to every r members of a family of $\mathcal{A}$ implies the existence of a k-tranversal to $r+1$ members of $\mathcal{A}$. They posed the following problems.

Problem 3.

- *For $r \geq 5$, do there exist integers n_r such that: If $\mathcal{A}$ is a finite family of at least n_r convex sets in $\mathbb{R}^3$ and every r members of $\mathcal{A}$ have a line transversal, then some $r+1$ members of $\mathcal{A}$ have a line transversal? (More generally, if $\mathcal{A}$ is in $\mathbb{R}^d$ and every r members of $\mathcal{A}$ have a k-transveral, does some set of $r+1$ members of $\mathcal{A}$ have a k-transversal?)*

- *For $p \geq q \geq 5$, do there exist integers $c_{p,q}$ such that: If $\mathcal{A}$ is a finite family of at least p convex sets in $\mathbb{R}^3$ and out of every p members of $\mathcal{A}$ some q have a line transversal, then some set of $c_{p,q}$ lines intersects every member of $\mathcal{A}$? (More generally, if $\mathcal{A}$ is in $\mathbb{R}^d$ and out of every p members of $\mathcal{A}$ some q have a k-transveral, does some set of $c_{p,q,k,d}$ k-flats intersects every member of $\mathcal{A}$?)*

6 Combinatorial Complexity

Let $\mathcal{A}$ be a family of convex sets in $\mathbb{R}^d$. Much of the early work on transversals was motivated by Helly's theorem and directed toward giving conditions for the existence of a k-transversal of $\mathcal{A}$. More recently, researchers have studied the structure of $\mathcal{T}_k^d(\mathcal{A})$, the space of all k-transversals of $\mathcal{A}$. Computer scientists, in particular, have been interested in explicitly constructing representations of this set.

The boundary of $\mathcal{T}_k^d(\mathcal{A})$ consists of k-transversals which are tangent to one or more members of $\mathcal{A}$. A *face* of this boundary is a maximally connected region of k-flats which are tangent to the same members of $\mathcal{A}$. The *combinatorial complexity* of $\mathcal{T}_k^d(\mathcal{A})$ is the number of such faces. A major problem has been to bound this combinatorial complexity for various families of convex sets, particularly convex sets bounded by algebraic surfaces, and thus bound the size of an explicit representation of $\mathcal{T}_k^d(\mathcal{A})$.

Most progress has been made in bounding the complexity of $\mathcal{T}_{d-1}^d(\mathcal{A})$, the space of hyperplane transversals to $\mathcal{A}$. Let $\mathcal{D}(h)$ be the mapping which takes each "non-vertical" hyperplane

$$h = \{(x_1, \ldots, x_d) : x_d = \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_{d-1} x_{d-1} + \sigma_d\}.$$

to the point $(\sigma_1, \ldots, \sigma_d)$ in dual space. The set of non-vertical hyperplane transversals to $\mathcal{A}$ is represented by a set of points in this dual space. (See Figure 1.)

To construct this set in dual space, consider a single compact convex set $a \in \mathcal{A}$. The non-vertical hyperplanes tangent to set a map to two unbounded surfaces in the dual space. The hyperplanes intersecting set a map to the points between these two surfaces. (See Figure 2.)

More precisely, define

$$\begin{aligned} \phi_a^+(\sigma_1, \ldots, \sigma_{d-1}) &= \max\{\sigma_d : \mathcal{D}^{-1}(\sigma_1, \ldots, \sigma_{d-1}, \sigma_d) \cap a \neq \emptyset\} \text{ and} \\ \phi_a^-(\sigma_1, \ldots, \sigma_{d-1}) &= \min\{\sigma_d : \mathcal{D}^{-1}(\sigma_1, \ldots, \sigma_{d-1}, \sigma_d) \cap a \neq \emptyset\}. \end{aligned}$$

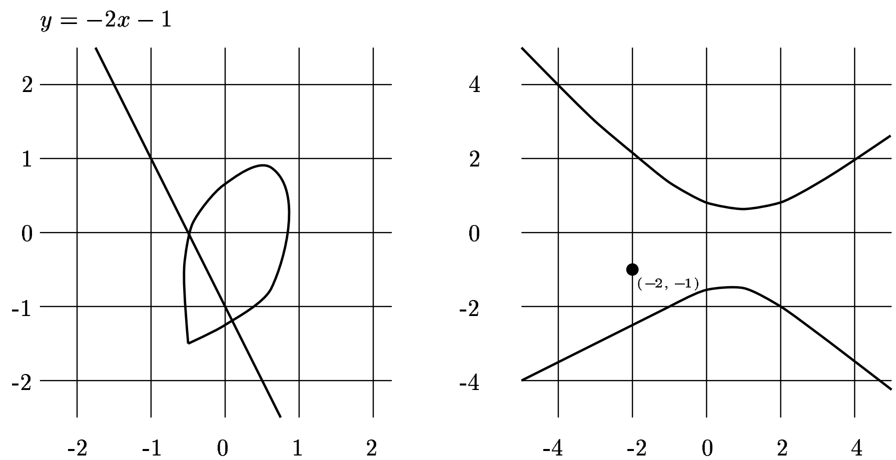

Figure 2: A convex set and its transversals.

For fixed $\{\sigma_1, \ldots, \sigma_{d-1}\}$ the set

$$\{\mathcal{D}^{-1}(\sigma_1, \ldots, \sigma_{d-1}, \sigma_d) : \phi_a^+(\sigma_1, \ldots, \sigma_{d-1}) \le \sigma_d \le \phi_a^-(\sigma_1, \ldots, \sigma_{d-1})\}$$

is a set of parallel hyperplanes which intersect a.

For each $a \in \mathcal{A}$, the graphs of functions ϕ_a^+ and ϕ_a^- define two surfaces in $\mathbb{R}^d$. The "corridor" of points between those surfaces represents the set of hyperplanes intersecting a. (If a is not closed, the boundaries of the corridors may not represent hyperplanes intersecting a. If a is not bounded, the functions ϕ_a^+ and ϕ_a^- may not be defined at all points.) The intersection of all these corridors represents the set of non-vertical hyperplane transversals to $\mathcal{A}$. Since each corridor is bounded by a "top" and "bottom" surface, the points corresponding to hyperplane transversals are the points below all the "top" surfaces and above all the "bottom" surfaces. Formally, the space of non-vertical hyperplane transversals to a family $\mathcal{A}$ of compact convex sets is

$$\{\mathcal{D}^{-1}(\sigma_1, \ldots, \sigma_{d-1}, \sigma_d) : \min_{a \in \mathcal{A}} \phi_a^+(\sigma_1, \ldots, \sigma_{d-1}) \ge \sigma_d \ge \max_{a \in \mathcal{A}} \phi_a^-(\sigma_1, \ldots, \sigma_{d-1})\}.$$

The function $\min_{a \in \mathcal{A}} \phi_a^+(\sigma_1, \ldots, \sigma_{d-1})$ is called the *lower envelope* of the functions $\{\phi_a^+\}$. Similarly, $\max_{a \in \mathcal{A}} \phi_a^-(\sigma_1, \ldots, \sigma_{d-1})$ is the *upper envelope* of the functions $\{\phi_a^-\}$. A representation of the space of non-vertical hyperplane transversals of $\mathcal{A}$ can be constructed by computing the lower and upper envelopes of $\{\phi_a^+\}$ and $\{\phi_a^-\}$, respectively, and then intersecting these envelopes.

Upper and lower envelopes have been studied extensively over the past fifteen years, particularly by Micha Sharir and his students and colleagues who made great progress in bounding their combinatorial complexity. This work led to asymptotically tight bounds in 1989 by Pach and Sharir on the combinatorial complexity of upper and lower envelopes of piecewise linear functions.

Edelsbrunner, Guibas and Sharir applied these bounds to the combinatorial complexity of the space of hyperplane transversals to a family of convex polytopes:

Theorem 12 [56, 23]. *Let $\mathcal{A}$ be a family of convex polytopes in $\mathbb{R}^d$ with a total of n_f faces. The combinatorial complexity of the space of hyperplane transversals to $\mathcal{A}$ is $O(n_f^{d-1}\alpha(n))$.*

$\alpha(n)$ is the slow growing inverse of the Ackermann function.

Let $\lambda_s(n)$ be the maximum length of an (n, s) Davenport-Schinzel sequence. (See [60] for the definition and discussion of Davenport-Schinzel sequences and Ackermann's function and their relationship to upper and lower envelopes.) $\lambda_s(n)$ is almost but not quite linear in n and is $O(n\alpha(n)^{O(\alpha(n)^{(s-2)/2})})$. Atallah and Bajaj bounded the complexity of line transversals in the plane in terms of $\lambda_s(n)$.

Theorem 13 [10]. *Let $\mathcal{A}$ be a family of n compact connected sets in $\mathbb{R}^2$ such that any two members of $\mathcal{A}$ have at most s common supporting lines. The combinatorial complexity of the space of line transversals to $\mathcal{A}$ is $O(\lambda_s(n))$.*

In $\mathbb{R}^3$, Agarwal, Schwarzkopf and Sharir bounded the complexity of the space of plane transversals to convex sets bounded by algebraic surfaces with degree less than or equal to some fixed constant.

Theorem 14 [3]. *Let $\mathcal{A}$ be a family of convex sets in $\mathbb{R}^3$ bounded by algebraic surfaces of bounded degree. The combinatorial complexity of the space of plane transversals to $\mathcal{A}$ is $O(n^{2+\epsilon})$, for any $\epsilon > 0$.*

Of course, the hidden constant in the O-notation depends upon ϵ and the maximum degree of the algebraic surfaces.

For hyperplane transversals to families of balls, the known bounds on the combinatorial complexity is drastically lower.

Theorem 15 [37]. *Let $\mathcal{A}$ be a family of n $(d–1)$-balls in $\mathbb{R}^d$. The combinatorial complexity of the space of hyperplane transversals to $\mathcal{A}$ is $O(n^{\lceil d/2 \rceil})$.*

It is not known if this bound is asymptotically tight.

The proof by Houle et al. is based on the fact that a convex polytope in $\mathbb{R}^{d+1}$ which is the intersection of n half-spaces has $O(n^{\lceil d/2 \rceil})$ faces. Represent each hyperplane

$$h = \{(x_1, \ldots, x_d) : \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_d x_d + \sigma_{d+1} = 0\}$$

uniquely by the $(d+1)$-tuple $(\sigma_1, \ldots, \sigma_{d+1})$ where $\sigma_1^2 + \cdots + \sigma_d^2 = 1$. (Note this representation is different from the previous one.) The hyperplanes which intersect a ball centered at $(c_1, c_2, \ldots, c_d)$ with radius r are:

$$\{(\sigma_1, \ldots, \sigma_{d+1}) : r \geq (\sigma_1 c_1 + \cdots + \sigma_d c_d + \sigma_{d+1}) \geq -r \text{ and } \sigma_1^2 + \cdots + \sigma_d^2 = 1.\}$$

Thus each ball defines two linear inequalities or, equivalently, two half-spaces. The hyperplane transversals for n balls is the intersection of $2n$ half-spaces, two for each ball, and the quadric surface $\sigma_1^2 + \cdots + \sigma_d^2 = 1$. The intersection of the $2n$ half-spaces has $O(n^{\lceil d/2 \rceil})$ faces and the intersection with the quadric surface has the same complexity.

Finally, some bounds are known for the space of line transversals in $\mathbb{R}^3$. A series of papers led to nearly tight bounds by Agarwal on the complexity of line transversals to convex polytopes in $\mathbb{R}^3$:

Theorem 16 [1]. *Let $\mathcal{A}$ be a family of convex polytopes in $\mathbb{R}^3$ with a total of n_f faces. The combinatorial complexity of the space of line transversals to $\mathcal{A}$ is $O(n_f^3 \log n)$.*

A very recent result by Agarwal, Aronov and Sharir bounds the complexity of the space of line transversals to balls in $\mathbb{R}^3$.

Theorem 17 [2]. *Let $\mathcal{A}$ be a family of n balls in $\mathbb{R}^3$. The combinatorial complexity of the space of line transversals to $\mathcal{A}$ is $O(n^{3+\epsilon})$, for any $\epsilon > 0$.*

In higher dimensions, there are no published tight or nearly tight bounds on the combinatorial complexity of the space of transversals.

Problem 4.

- *What is the combinatorial complexity of the space of k-transversals to a family of convex polytopes in $\mathbb{R}^d$ for $d \geq 4$ and $k < d-1$?*
- *What is the combinatorial complexity of the space of line transversals to a family of convex sets bounded by algebraic surfaces of bounded degree in $\mathbb{R}^3$ or the space of k-transversals in $\mathbb{R}^d$ for $d \geq 4$? (This is not even known for hyperplane transversals when $d \geq 4$.)*

Most of the above results bound the complexity of the space of transversals in terms of the complexity of the underlying objects, either the total number of polytope faces or the maximum degree of the algebraic curves bounding the convex sets. This is because even two convex sets in the plane could have an arbitrary number of common tangent lines, each of which is a face in the space of line transversals. Thus there is no finite bound which is only in terms of n on the complexity of the space of k-transversals to an arbitrary family of n convex sets. However, if $\mathcal{A}$ is a family of pairwise disjoint compact convex sets in the plane, then any two sets in $\mathcal{A}$ have exactly four common tangent lines. By Theorem 13, the combinatorial complexity of the space of line transversals to $\mathcal{A}$ is $O(\lambda_4(n))$. In fact, a closer analysis gives the combinatorial complexity as $O(n)$.

Two sets are pairwise disjoint if they have no point transversal. The proper generalization for m sets in higher dimensions is that they have no $(m-2)$-transversal. In this case, in place of four tangent lines to two convex sets, Cappell et. al. showed that the space of tangent hyperplanes to $m \leq d$ compact strictly convex sets in $\mathbb{R}^d$ form 2^{m-1} copies of $\mathbb{S}^{d-m}$, the $(d-m)$-sphere.

Theorem 18 [12]. *Let $\mathcal{A}$ be an $(m-2)$-separated family of $m \leq d$ compact strictly convex sets in $\mathbb{R}^d$. The space of hyperplanes simultaneously tangent to each member of $\mathcal{A}$ is homeomorphic to the union of 2^{m-1} copies of $\mathbb{S}^{d-m}$.*

The condition of strict convexity can be dropped when m equals d. It is not known if this condition is required for m less than d.

The proof of Theorem 18 involves shrinking the m convex sets in $\mathcal{A}$ to m points. The set of hyperplanes through $m \leq d$ points in general position is homeomorphic to $\mathbb{S}^{d-m}$. After some topological analysis, it follows that the set of hyperplane tangents which separate $\mathcal{A}$ in a particular manner is also homeomorphic to $\mathbb{S}^{d-m}$. Since there are 2^{m-1} ways in which hyperplanes separate $\mathcal{A}$ (including not separating $\mathcal{A}$ at all) the space of hyperplane transversals is homeomorphic to 2^{m-1} copies of $\mathbb{S}^{d-m}$. Lewis, Von Hohenbalken and Klee [51] gave a simple, elegant proof for the case m equal to d using Brouwer's fixed point theorem.

Theorem 18 implies that the hyperplane tangents to separated families behave essentially like topological spheres. By analyzing the complexity of the boundary of the union of a set of topological spheres, Cappell et. al. gave the following bound:

Theorem 19 [12]. *Let $\mathcal{A}$ be a $(d-2)$-separated family of n compact strictly convex sets in $\mathbb{R}^d$. The combinatorial complexity of the space of hyperplane transversals to $\mathcal{A}$ is $O(n^{d-1})$.*

The only known lower bounds are for line transversals in the plane in which case the bound is tight.

No bounds are known on the complexity of the space of k-transversals to a $(k-1)$-separated family of convex sets in $\mathbb{R}^d$. Anderson noted that the space of k-flats which are simultaneously tangent to a $(k-1)$-separated family of $k+1$ balls in $\mathbb{R}^d$ is $(S^{d-k-1})^{k+1}$, the product of $k+1$ spheres of dimension $d-k-1$ [8].

Problem 5. *Let $\mathcal{A}$ be a $(k-1)$-separated family of compact convex sets in $\mathbb{R}^d$.*

- *What is the topological structure of the space of k-flats simultaneously tangent to $m \leq k+1$ members of $\mathcal{A}$?*
- *What is the combinatorial complexity of the space of k-transversals to $\mathcal{A}$?*

7 Counting Geometric Permutations

Let $\mathcal{A}$ be a $(k-1)$-separated family of convex sets in $\mathbb{R}^d$. Two k-transversals which induce different pairs of oriented matroids on $\mathcal{A}$ must lie in different connected components of $\mathcal{A}$. Thus a lower bound on the number of pairs of oriented matroids induced by k-transversals of $\mathcal{A}$ is also a lower bound on the number of connected components of $\mathcal{A}$. When k equals $d-1$, each connected component is associated with a unique pair of oriented matroids on $\mathcal{A}$ induced by a hyperplane transversal of $\mathcal{A}$. The number of connected components equals the number of such pairs. Given a $(k-1)$-separated family of convex sets in $\mathbb{R}^d$, how many different pairs of oriented matroids on $\mathcal{A}$ are induced by k-transversals of $\mathcal{A}$?

In 1985, Katchalski, Lewis and Zaks [47] constructed families of $n \geq 4$ pairwise disjoint convex sets in $\mathbb{R}^2$ which have $2n-2$ geometric permutations. Five years later, Edelsbrunner and Sharir showed that $2n-2$ was the maximum achievable.

Theorem 20 [24]. *Let $\mathcal{A}$ be a family of $n \geq 4$ pairwise disjoint compact convex sets in $\mathbb{R}^2$. The maximum number of pairs of linear orderings induced by line transversals of $\mathcal{A}$ is $2n-2$.*

Represent the directions in the plane as points on the unit circle. Edelsbrunner and Sharir noted that each directed line transversal can can be translated to the right until it is tangent to at least one convex set $a \in \mathcal{A}$. Label the direction of that line transversal "a". This labelling divides the transversal directions into labelled arcs which form a cyclic sequence around the unit circle. This cyclic sequence has the property that $a \ldots b \ldots a \ldots b \ldots$ is not a subcycle for any $a, b \in \mathcal{A}$. These sequences are $(n, 2)$ Davenport-Schinzel cycles and their maximum length is $2n-2$. (See [60, p.7].) Since each geometric permutation corresponds to one or more arcs, $2n-2$ is also a bound on the number of geometric permutations.

Any bound on the combinatorial complexity of the space of k-transversals to $\mathcal{A}$ is also a bound on the number of oriented matroids induced by k-transversals of $\mathcal{A}$. In particular, Theorem 18 implies the following:

Theorem 21 [12]. *Let $\mathcal{A}$ be a $(d-2)$-separated family of n convex sets in $\mathbb{R}^d$. The maximum number of oriented matroids induced by oriented hyperplane transversals of $\mathcal{A}$ is $O(n^{d-1})$.*

As with Theorem 18, it is not known if this bound is tight.

Finally, Goodman, Pollack and Wenger bounded the number of oriented matroids induced by k-transversals in $\mathbb{R}^d$ for general k and d.

Theorem 22 [31]. *Let $\mathcal{A}$ be a $(k-1)$-separated family of n compact convex sets in $\mathbb{R}^d$ where $k \leq d-2$. The maximum number of oriented matroids induced by oriented k-transversals of $\mathcal{A}$ is $O(n^{k(k+1)(d-k)})$.*

The $O(n^{k(k+1)(d-k)})$ term actually bounds the maximum size of a set of incompatible oriented matroids induced by k-transversals to subsets of $\mathcal{A}$ and is probably not tight. Two oriented matroids of two possibly different subsets $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{A}$ are *incompatible* if the two oriented matroids restricted to their common elements, $\mathcal{A}' \cap \mathcal{A}''$, are different. Note also that when k equals $d-1$, the $O(n^{k(k+1)(d-k)})$ term evaluates to $O(n^{(d-1)d})$ which does not match the $O(n^{d-1})$ term in Theorem 21.

The order in which a directed line l intersects two sets separated by a hyperplane h depends upon the order in which l intersects the two half-spaces defined by h. For each pair of convex sets $a, a' \in \mathcal{A}$, choose a hyperplane strictly separating a from a''. A careful analysis of the number of different ways a line can intersect these $\binom{n}{2}$ hyperplanes gives an $O(n^{2(d-1)})$ bound on the maximum number of oriented matroids induced by line transversals of $\mathcal{A}$.

Similarly, a family $\mathcal{A}$ of compact convex sets is $(k-1)$-separated if and only if every j sets in $\mathcal{A}$ can be strictly separated by a hyperplane from every other $k+1-j$ sets. The oriented matroid (consisting of a single orientation) induced by an oriented k-transversal on $k+1$ sets is determined by the manner in which the k-transversal intersects these separating hyperplanes, i.e., the $2^k - 1$ hyperplanes which separate every j sets from every other $k+1-j$ sets for every $1 \leq j \leq k$. In fact, the oriented matroid depends only on the relationship between the k-transversals and the normal vectors to these hyperplanes. For each pair of subsets $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{A}$ of size j and $k+1-j$, respectively, choose a hyperplane strictly separating $\mathcal{A}'$ and $\mathcal{A}''$. A careful analysis of the number of different ways a k-flat can intersect these $(2^k - 1)\binom{n}{k+1}$ hyperplanes gives the $O(n^{k(k+1)(d-k)})$ bound in Theorem 22.

The only lower bounds known are by Katchalski, Lewis and Liu for line transversals in $\mathbb{R}^d$.

Theorem 23 [46]**.** *There exist families $\mathcal{A}$ of n pairwise disjoint convex sets in $\mathbb{R}^d$ whose directed line transversals induce $\Omega(n^{d-1})$ linear orderings of $\mathcal{A}$.*

Even for line transversals in $\mathbb{R}^3$, the asymptotic upper and lower bounds, $O(n^4)$ and $\Omega(n^2)$, do not match.

Problem 6. *Give asymptotically tight upper and lower bounds on the maximum number of linear orderings induced by line transversals of n pairwise disjoint convex sets in $\mathbb{R}^3$ (or more generally give tight bounds on the maximum number of oriented matroids induced by k-transversals of $(k-1)$-separated convex sets in $\mathbb{R}^d$.)*

8 Convexity on the Affine Grassmannian

In a 1995 paper entitled "Foundations of a theory of convexity on affine Grassmannian manifolds" [28], Goodman and Pollack suggest yet another way of exploring geometric transversals. Instead of asking for conditions for the existence of transversals or studying the structure of the space of transversals, they ask under what conditions is a set $\mathcal{F}$ of k-flats the space of transversals to some family of convex sets? They answer that $\mathcal{F}$ is a space of transversals to some family $\mathcal{A}$ of convex sets if and only if every k-flat "surrounded" by $\mathcal{F}$ is in $\mathcal{F}$. A k-flat f is *surrounded* by $\mathcal{F}$ if $f \in \mathcal{F}$ or there is some j-flat g containing f such that every $(j-1)$-flat in g which contains f strictly separates two k-flats of $\mathcal{F}$ lying in g; i.e., f is "trapped" by the elements of $\mathcal{F}$ lying in g. The family $\mathcal{A}$ of convex sets need not be finite and the convex sets in $\mathcal{A}$ may not be compact.

Given a family $\mathcal{F}$ of k-flats in $\mathbb{R}^d$, let $\mathcal{C}(\mathcal{F})$ be the set of all k-flats surrounded by $\mathcal{F}$ or, alternatively, the smallest space of k-transversals containing $\mathcal{F}$. $\mathcal{C}(\mathcal{F})$

shares many properties with the convex hull of a set of points. For one thing, if $\mathcal{F}$ is a family of 0-flats or points, then $\mathcal{C}(\mathcal{F})$ is simply the convex hull of the point set $\mathcal{F}$. As with convex hulls of point sets, the operator $\mathcal{C}$ is monotone (increasing) and idempotent, i.e.:

- $\mathcal{F} \subseteq \mathcal{C}(\mathcal{F})$;
- $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow \mathcal{C}(\mathcal{F}_1) \subseteq \mathcal{C}(\mathcal{F}_2)$; and
- $\mathcal{C}(\mathcal{C}(\mathcal{F})) = \mathcal{C}(\mathcal{F})$.

It also has an "anti-exchange" property:

- If $f, f' \notin \mathcal{C}(\mathcal{F})$ and $f \in \mathcal{C}(\mathcal{F} \cup \{f'\})$ while $f' \in \mathcal{C}(\mathcal{F} \cup \{f\}$, then $f = f'$.

Finally, it is invariant under nonsingular affine transformations, so

- $\mathcal{C}(\sigma(\mathcal{F})) = \sigma(\mathcal{C}(\mathcal{F}))$ for nonsingular affine transformations σ.

$\mathcal{C}$ also satisfies a property similar to the Krein-Milman property for point sets. The *extreme* k-flats of $\mathcal{F}$, denoted $\mathrm{ext}(\mathcal{F})$, are the k-flats $f \in \mathcal{F}$ which are not surrounded by $\mathcal{F} \setminus \{f\}$, i.e., $f \notin \mathcal{C}(\mathcal{F} \setminus \{f\})$.

- If $\mathcal{F} = \mathcal{C}(\mathcal{F})$ and $\mathcal{F}$ is compact, then $\mathcal{F} = \mathcal{C}(\mathrm{ext}\,\mathcal{F})$.

Because of these many similarities with point convexity, Goodman and Pollack called $\mathcal{C}$ the convex hull operator for the affine Grassmannian. A set of k-flats $\mathcal{F}$ is convex if $\mathcal{F} = \mathcal{C}(\mathcal{F})$, or, equivalently, if $\mathcal{F}$ is the space of k-transversals to some family of convex point sets. However, there is one major difference between convex point sets and convex sets of k-flats. Convex point sets are always connected while a space of k-transversals may have many connected components. Goodman and Pollack showed that this difference is inevitable since there is no way of defining convex sets of k-flats which have the properties listed above and are always connected.

As we noted, $\mathcal{C}(\mathcal{F})$ is not necessarily connected. However, if $\mathcal{F}$ is connected, is $\mathcal{C}(\mathcal{F})$ connected? Equivalently, if $\mathcal{F}$ is convex (i.e., $\mathcal{F} = \mathcal{C}(\mathcal{F})$) is every connected component of $\mathcal{F}$ convex? Goodman, Pollack and Wenger showed this was false by constructing a connected set $\mathcal{F}$ of lines parallel to the (x, y)-plane in $\mathbb{R}^3$ which surround the z-axis and no other line. $\mathcal{C}(\mathcal{F})$ equals $\mathcal{F} \cup \{z\text{-axis}\}$ which is clearly not connected. Alternatively, the connected component $\mathcal{F}$ of $\mathcal{C}(\mathcal{F})$ is not convex, since $\mathcal{F} \neq \mathcal{C}(\mathcal{F})$.

The set $\mathcal{F}$ given by Goodman et. al. was not closed. What if $\mathcal{F}$ is both closed and connected? What if it is compact and connected? If $\mathcal{F}$ is compact and convex, is every connected component of $\mathcal{F}$ convex? Goodman, Pollack and Wenger showed that the last statement was true for lines in $\mathbb{R}^3$ under stronger conditions.

Theorem 24 [30]**.** *Let $\mathcal{A}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^3$. If $\mathcal{L}$ is a connected component of $\mathcal{T}_1^3(\mathcal{A})$, then $\mathcal{L} = \mathcal{C}(\mathcal{L})$. Moreover, $\mathcal{L}$ is itself the space of line transversals to some finite family of pairwise disjoint compact convex point sets.*

Goodman et. al. first proved that every two lines in $\mathcal{C}(\mathcal{L}) \subseteq \mathcal{T}_1^3(\mathcal{A})$ induce the same geometric permutation on $\mathcal{A}$. They then showed that if two line transversals which induce the same geometric permutation of $\mathcal{A}$ intersect, then the lines lie in the same connected component of $\mathcal{T}_1^3(\mathcal{A})$. Finally, they proved that if l is some line in $\mathcal{C}(\mathcal{L})$, then l is connected to a parallel line in $\mathcal{C}(\mathcal{L})$ which intersects a line in $\mathcal{L}$. Thus $\mathcal{C}(\mathcal{L})$ is connected and hence equals $\mathcal{L}$.

Problem 7.

- *Is it true that if $\mathcal{A}$ is a finite 1-separated family of compact convex sets in $\mathbb{R}^4$ and $\mathcal{F}$ is a connected component of $\mathcal{T}_2^4(\mathcal{A})$, then $\mathcal{F} = \mathcal{C}(\mathcal{F})$? (More generally, if $\mathcal{A}$ is $(k-1)$-separated and $\mathcal{F}$ is a connected component of $\mathcal{T}_k^d(\mathcal{A})$, then is $\mathcal{F} = \mathcal{C}(\mathcal{F})$?)*
- *Is it true that if $\mathcal{F}$ is a compact, connected family of lines in $\mathbb{R}^3$, then $\mathcal{C}(\mathcal{F})$ is connected, (or, more generally, that $\mathcal{C}(\mathcal{F})$ is connected for any compact, connected family of k-flats?)*

9 Surveys

The classical 1963 survey "Helly's Theorem and its relatives" by Danzer, Grünbaum and Klee [14] contains much of the early work on geometric transversal theory. Two recent survey papers, "Helly, Radon and Carathéodory type theorems" by Eckhoff [22] and "Geometric transversal theory" by Goodman, Pollack and Wenger [29] describe the progress made in the eighties. The *Handbook of Discrete and Computational Geometry* contains a chapter "Helly-Type Theorems and Geometric Transversals" by Wenger [67]. Much of the material on the combinatorial complexity of the space of transversals can be found in Sharir and Agarwal's book *Davenport–Schinzel Sequences and Their Geometric Applications* [60], including definitions and analyses of Davenport-Schinzel sequences and upper and lower envelopes.

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