A 2D Kinetic Triangulation with Near-Quadratic Topological Changes *

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ABSTRACT

Given a set of n points S in the plane, a *triangulation* of S is a subdivision of the convex hull into triangles whose vertices are from S. In the kinetic setting, the input point set is replaced by a continuous family $S(t)$ indexed by time t. We wish to maintain a triangulation for $S(t)$ with small number of topological events, i.e., small number of changes (such as insertion or deletion of an edge) to the triangulation through time. In particular, we propose a kinetic data structure (KDS) that processes $O(n^2 2^{O(\sqrt{\log n \cdot \log \log n})})$ topological events with high probability if the trajectories of input points are algebraic curves of fixed degree. The total time for maintaining this triangulation using KDS is of roughly the same order. Our algorithm relies on a *hierarchical fan triangulation* that we propose, which is built based upon a randomized hierarchical scheme. This is the first known KDS for maintaining a triangulation that processes near-quadratic events. It improves over the $O(n^{7/3})$ previous best known result [2], and almost matches the $\Omega(n^2)$ lower bound [1]. The number of events can be reduced to $O(nk \cdot 2^{O(\sqrt{\log k \cdot \log \log n})})$ if only k of the points are moving.

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Geometrical problems and computations

General Terms

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Keywords

triangulation, kinetic data structures

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1. INTRODUCTION

Let S be a set of n points in $\mathbb{R}^2$. A *triangulation* of S is a subdivision of the convex hull into triangles whose vertices are from S. Motivated by applications in several areas, including computer graphics, physical simulation, collision detection, and geographic information systems, triangulations have been widely studied in computational geometry and the above areas [9, 12]. With the advancement in technology, many applications, for instance, video games, virtual reality, dynamic simulations, and robotics call for maintaining a triangulation as the points move. For example, the Arbitrary Eulerian-Lagrangian method [11] provides a way to integrate the motion of fluids and solids within a moving finite-element mesh. The time axis is discretized and the mesh vertices are moved between each time step so as to respect the interfaces between the different media. However, numerical problems arise when the mesh becomes too distorted, and the mesh generated depend on the discretization of time. Another approach to build more adaptive triangulations for such problems is to work in the space-time domain [13].

In the kinetic set-up, the input point set S is replaced by a continuous family $S(t)$ indexed by time. As $S(t)$ changes continuously, any fixed triangulation of S also deforms continuously. However, a triangulation computed for the initial point set $S(0)$ cannot be guaranteed to remain valid all the time, and it becomes necessary to *update* a triangulation over time by deleting some of the existing edges and (re-)inserting some other edges. We refer to each of such insertion or deletion in the triangulation as an *topological event*. We expect that the topological events occur only at discrete time instances. In this paper we study how to maintain a triangulation so that the number of topological events is as small as possible.

1.1 Related Work

Since Atallah [7] initiated the study of geometric problems involving moving objects, much work has been devoted to this area due to its importance in both theory and applications of computational geometry; see [3, 5, 14] for reviews on kinetic geometric algorithms and data structures. The early work on kinetic geometry mostly focused on bounding the number of combinatorial or topological changes in various geometric structures as the input objects move. Later

Basch *et al.* [8] introduced a general framework, the so-called *kinetic data structure* (KDS), for maintaining a discrete attribute of objects in predictable motion. Their approach to maintain a given attribute $A(t)$ for a continuously changing scene $S(t)$ is as follows: at a given time t, we create a proof of correctness of the attribute based on elementary tests called *certificates*. For each certificate, we compute the time at which it fails and put it in a global event queue. As the attribute cannot change while all tests remain valid, it is unnecessary to perform any computation until the first certificate fails. When a certificate fails (an *event*), the discrete attribute is updated if it needs to be, and a new proof of correctness is constructed by making some modifications to the previous proof of correctness. Their approach led to efficient algorithms for several kinetic problems [14].

In the context of triangulation, a longstanding open problem is to bound the number of topological events in the Delaunay triangulation of a set of moving points in $\mathbb{R}^2$. The best known upper bound is near-cubic if trajectories of input points are algebraic curves of fixed degree, and the bound is cubic if the trajectories are linear [6]. Although it is conjectured that the number of topological changes is $O(n^2)$, no such bound is known even for maintaining an arbitrary triangulation of a set of moving points in $\mathbb{R}^2$. Agarwal *et al.* [2] described a scheme for maintaining a triangulation of a set of points that incurs roughly $n^{7/3}$ topological changes if the points are moving linearly. Chew [10] proved that the L_1 Delaunay triangulation of S changes $O(n^2\alpha(n))$ times, where $\alpha(n)$ is the inverse Ackermann function; however, the L_1 Delaunay triangulation is not necessarily a triangulation of the convex hull of the point set.

Agarwal *et al.* [1] showed that the convex hull of S may change $\Theta(n^2)$ times if the points are moving linearly; this result immediately implies a lower bound $\Omega(n^2)$ on the number of topological changes to any triangulation. This lower bound holds even if linear number of Steiner points (not necessarily inside $\mathrm{conv}(S)$) are allowed [4].

1.2 Our Results

Let $S = \{p_1, \dots, p_n\}$ be a set of points in $\mathbb{R}^2$, and let $p_i(t) = (x_i(t), y_i(t))$ denote the position of p_i at time t, and let $S(t)$ denote the configuration of S at time t. We assume that $x_i(\cdot), y_i(\cdot)$ are polynomials of fixed degree. We describe a kinetic data structure that processes $O(n^2 2^{O(\sqrt{\log n \cdot \log\log n})})$ topological events, each of which is insertion or deletion of an edge. The complexity of the KDS, i.e. the total time spent in maintaining the triangulation, is of the same order. This is the first known kinetic data structure for maintaining a triangulation that processes near-quadratic topological events. The number of events can be reduced to $O(nk \cdot 2^{O(\sqrt{\log k \cdot \log\log n})})$ if only k of the points are moving.

Our algorithm relies on a randomized hierarchical scheme. We first describe the so-called fan triangulation (Section 2), originally introduced in [2], and then introduce the notion of constrained fan triangulation with respect to a planar subdivision (Section 3). We choose a random sample $R \subseteq S$, compute a triangulation of R recursively, and then compute the constrained fan triangulation of S with respect to the triangulation computed for R (Section 4). We analyze the events in constrained fan triangulation and show that if R is a random subset of appropriate size, the total number of events is near-quadratic (Section 5).

2. FAN TRIANGULATION

Let $S = \{p_1, \cdots, p_n\}$ be a set of n points in $\mathbb{R}^2$, sorted in non-increasing order by their y-coordinates, i.e., $y(p_1) \geq y(p_2) \dots \geq y(p_n)$. For a point $q \in \mathbb{R}^2$, let $S_q = \{p_i \in S \mid y(p_i) > y(q)\}$. Denote by $\mathcal{V}(q)$ the set of vertices from $\partial\,\mathrm{conv}(S_q)$ that are visible from q, i.e., $p_i \in \mathcal{V}(q)$ if the relative interior of the segment qp_i does not intersect $\mathrm{conv}(S_q)$. Furthermore, let $\rho(q)$ (resp. $\gamma(q)$) denote the point from $\mathcal{V}(q)$ such that the oriented line $\overrightarrow{q\rho(q)}$ (resp. $\overrightarrow{q\gamma(q)}$) is the left (resp. right) tangent of $\mathrm{conv}(S_q)$. Obviously, $\mathcal{V}(q)$ is the subset of vertices on $\partial\,\mathrm{conv}(S_q)$ lying between $\rho(q)$ and $\gamma(q)$, assuming that the vertices are ordered in counterclockwise direction.

The *fan triangulation* of S is constructed by sweeping a horizontal line h from $y = +\infty$ to $y = -\infty$. At any time the algorithm maintains the fan triangulation of points from S that lie above h. It updates the triangulation when the sweep line crosses a point $p_i \in S$ by adding the edges p_ip_j for all $p_j \in \mathcal{V}(p_i)$; see Figure 1. The triangulation at the end of the sweep is the fan triangulation of S, which we denote as $\mathcal{F}(S)$.

We classify the edges of $\mathcal{F}(S)$ incident upon a point $p_i \in S$ into two classes:

- (i) *up edges*: edges p_ip_j so that $j < i$; p_j is also referred to as an *up neighbor* of p_i.
- (ii) *down edges*: edges p_ip_j so that $j > i$; p_j is also called a *down neighbor* of p_i. Furthermore, if $p_i = \rho(p_j)$ or $p_i = \gamma(p_j)$, then edge p_ip_j is referred to as a *convex edge*, otherwise, it is a *reflex edge*.

The following properties of $\mathcal{F}(S)$ are straightforward to prove.

- (F1) For $1 \leq i < n$, p_ip_{i+1} is an edge of $\mathcal{F}(S)$.
- (F2) For each point p_i, at most one of its down edges is a reflex edge. In particular, let $p_ip_{j_1}, p_ip_{j_2}, \cdots, p_ip_{j_k}$ be the sequence of the down edges incident upon p_i, with $j_1 < \cdots < j_k$. Then either $p_i = \rho(p_{j_1})$ and the sequence of edges is sorted in clockwise direction, or $p_i = \gamma(p_{j_1})$ and the sequence is sorted in counterclockwise direction. Furthermore, in the former (resp. latter) case, if $p_ip_{j_k}$ is not a reflex edge, then $p_i = \rho(p_{j_k})$ (resp. $p_i = \gamma(p_{j_k})$).

Now suppose we are given a set of moving points $S(t) = \{p_1(t), p_2(t), \dots, p_n(t)\}$ in the plane, and we wish to maintain $\mathcal{F}(t) = \mathcal{F}(S(t))$ for any $t \in \mathbb{R}$. As the points from S move, $\mathcal{F}(S(t))$ deforms continuously. But the topology of $\mathcal{F}(S)$ changes only at discrete times, which we refer to as *events* (note the difference between events and topological events: an event may cause multiple topological events to the triangulation). As observed in [2], there are two types of events.

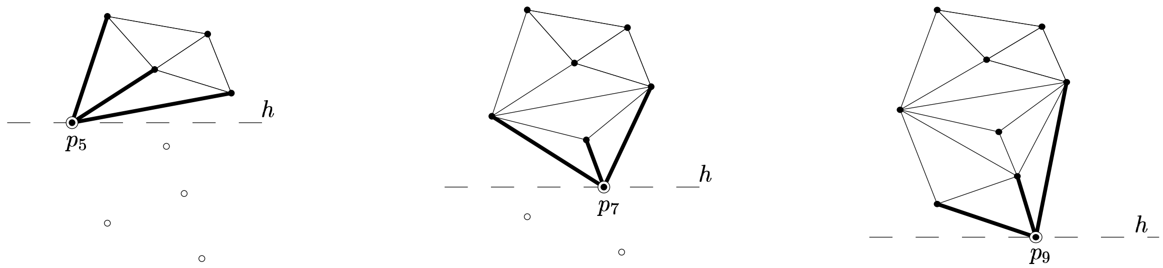

Figure 1: Construction of fan triangulation at various stages — the point denoted by double circle is being inserted, and the thick lines are added.

Ordering event. The y-coordinates of two points p_i and p_j become equal at time t_0. Assume w.o.l.g. that $y(p_i(t_0^-)) > y(p_j(t_0^-))$ and $y(p_i(t_0^+)) < y(p_j(t_0^+))$, where t_0^- (resp. t_0^+) is the time immediately before (resp. after) t_0. By property (F1), $p_i(t_0^-)p_j(t_0^-)$ (resp. $p_j(t_0^+)p_i(t_0^+)$) exists at time t_0^- (resp. t_0^+). In fact, both of them are necessarily convex edges; assume $p_i(t_0^-) = \rho(p_j(t_0^-))$ and $p_j(t_0^+) = \gamma(p_i(t_0^+))$. Let $p_{k_1}, p_{k_2}, \cdots, p_{k_u}$ be the sequence of vertices in $\mathcal{V}_{ij}(t_0) = \mathcal{V}(p_i(t_0^-)) \cap \mathcal{V}(p_j(t_0^+))$ ordered in counterclockwise direction. Obviously, at time t_0^-, $p_{k_l}p_i$, for all $1 \le l \le u$, and $p_{k_u}p_j$ are edges in $\mathcal{F}(t_0^-)$, while at time t_0^+, $p_{k_l}p_j$, for all $1 \le l \le u$, and $p_{k_1}p_i$ are edges in $\mathcal{F}(t_0^+)$. So to update the triangulation, we delete the edges $p_{k_2}p_i, \cdots, p_{k_u}p_i$ and insert the edges $p_{k_1}p_j, \cdots, p_{k_{u-1}}p_j$ to obtain $\mathcal{F}(t_0^+)$. See Figure 2 (a). The time spent is $O(|\mathcal{V}_{ij}(t_0)|)$.

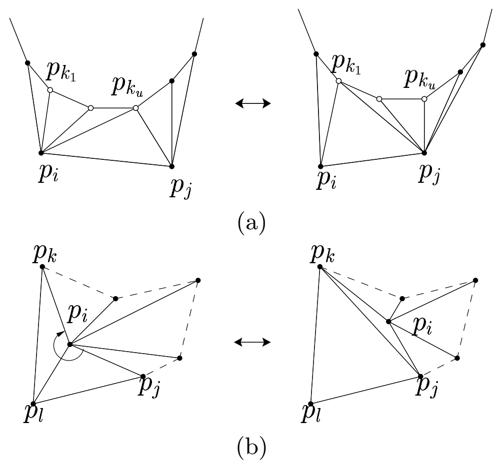

Figure 2: (a) Ordering event. Empty points are from $\mathcal{V}_{p_ip_j}(t_0)$. (b) Visibility event.

Visibility event. $\rho(p_j)$ (or $\gamma(p_j)$) changes for some point $p_j \in S$. Suppose that $p_i(t_0^-) = \rho(p_j(t_0^-))$ and $p_k(t_0^+) = \rho(p_j(t_0^+))$ with $y(p_k(t_0)) > y(p_i(t_0))$. Then $p_i(t_0)$, $p_j(t_0)$, and $p_k(t_0)$ are collinear. Furthermore, among all the convex edges incident upon p_i at t_0^-, p_ip_j is the leftmost edge. It then follows from property (F2) of the fan triangulation, that there is at most one edge (the reflex edge), say p_ip_l, between p_ip_j and p_ip_k in clockwise order around p_i. If p_ip_l does not exist, then p_i, p_j, and p_k are collinear on $\partial\,\mathrm{conv}(\mathcal{F}(t_0))$. To update the fan triangulation, we delete the edge p_ip_l (if it exists) and insert the edge p_jp_k (Figure 2 (b)). We thus spend $O(1)$ time at each visibility event. The case when $y(p_k(t_0)) < y(p_i(t))$ or when $\gamma(p_j)$ changes is symmetric.

In order to detect the above events, we maintain three families of certificates:

(i) For each edge $p_ip_j \in \mathcal{F}(t)$, the time t_0 at which $y(p_i(t_0)) = y(p_j(t_0))$.

(ii) For each triangle $p_ip_jp_k \in \mathcal{F}(t)$, with $y(p_j) < y(p_i) < y(p_k)$, the time for which $p_i(t_0) \in p_j(t_0)p_k(t_0)$.

(iii) For each point p_i, let p_j be the lowest such that $p_i = \rho(p_j)$ (resp. $p_i = \gamma(p_j)$) if it exists. We add the time at which p_j, p_i, and $\rho(p_i)$ (resp. p_j, p_i, and $\gamma(p_i)$) become collinear.

By property (F1), p_ip_j is an edge in the fan triangulation when p_i and p_j cause an ordering event, and by construction $p_i\rho(p_i)$, $p_i\gamma(p_i)$ are edges in the fan triangulation. Hence it is easy to verify that the above certificates detect all events.

3. CONSTRAINED FAN TRIANGULATION

In this section we introduce the notion of *constrained fan triangulation* and show how to maintain it under motion. As earlier, let $S = \{p_1, \cdots, p_n\}$ be a set of n points in $\mathbb{R}^2$ sorted in non-increasing order of their y-coordinates, and let Π be a set of segments with pairwise-disjoint interiors, and whose endpoints lie in S. We will be mostly interested in the case in which Π is a triangulation of a subset $R \subseteq S$, but we give the definition for the general setting.

Again, we construct the constrained fan triangulation of S by sweeping a horizontal line h from $y = +\infty$ to $y = -\infty$. Let $\mathcal{F}^{(0)} = \Pi$, and let $\mathcal{F}^{(i-1)}$ be the *partial* triangulation computed after the sweep line has processed p_{i-1}. If the interior of segment p_ip_j, for $j < i$, does not intersect $\mathcal{F}^{(i-1)}$, then p_ip_j is called *visible* with respect to $\mathcal{F}^{(i-1)}$. Define $\mathcal{V}(p_i)$, $\rho(p_i)$, and $\gamma(p_i)$ similarly as before, under the modified concept of visibility; $\rho(p_i)$ (resp. $\gamma(p_i)$) might be different with respect to different Π. When the sweep line crosses p_i, we compute $\mathcal{F}^{(i)}$ by adding the edges p_ip_j for all vertices p_j from $\mathcal{V}(p_i)$. Note that unlike in the sweeping process to construct the fan triangulation, $\mathcal{F}^{(i)}$ might not be a triangulation; it is possible that only one point from $p_1, \ldots, p_{i-1}$ is visible from p_i due to the constraint Π, in which case we add only one edge to $\mathcal{F}^{(i)}$, and this "dangling" edge of $\mathcal{F}^{(i)}$ will become part of a triangle at a later stage. The final triangulation $\mathcal{F}^{(n)}$ is the constrained fan triangulation of S (with respect to Π), denoted by $\mathcal{F}(S, \Pi)$. See Figure 3.

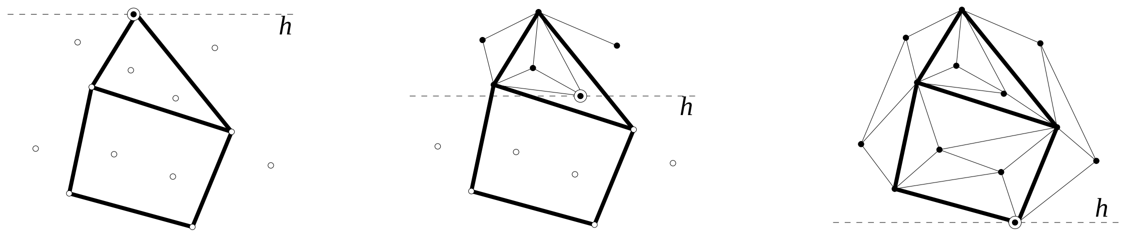

Figure 3: Constructing constrained fan triangulation w.r.t. Π (thick edges) at various stages.

Observe that if $\Pi = \partial\,\text{conv}(S)$, then the constrained fan triangulation of S (with respect to Π) is the same as the fan triangulation of S. If Π is a triangulation of a subset $R \subseteq S$, then Π partitions S into various subsets: $S_\Delta = S \cap \Delta$ (including vertices of Δ), where $\Delta \in \Pi$ or $\Delta = \Sigma$ is the region outside the convex hull of R. By the above observation, the constrained fan triangulation $\mathcal{F}(S, \Pi)$ restricted to S_Δ is the same as $\mathcal{F}(S_\Delta)$ for $\Delta \in \Pi$. Hence, $\mathcal{F}(S, \Pi)$ can be computed by constructing independently $\mathcal{F}(S_\Delta)$ for each $\Delta \in \Pi$ and by constructing $\mathcal{F}(S_\Sigma, \partial\,\text{conv}(R))$ within the exterior of conv(R), i.e. the constrained fan triangulation of S_Σ with respect to the boundary of conv(R).

The following properties of constrained fan triangulation are generalizations of (F1) and (F2).

(C1) If p_{i-1} is visible from p_i with respect to Π, then $p_{i-1}p_i$ is an edge in $\mathcal{F}(S, \Pi)$.

(C2) If $p_i p_{j_1}, \cdots, p_i p_{j_k}$ are the down edges incident upon p_i sorted in clockwise direction so that no edges of Π lie between them, then either $j_1 < \cdots < j_k$ and $p_i = \rho(p_{j_l})$ for $l < k$ ($l \leq k$ if $p_i p_{j_k}$ is not a reflex edge) or $j_1 > \cdots > j_k$ and $p_i = \gamma(p_{j_l})$ for $l < k$ ($l \leq k$ if $p_i p_{j_k}$ is not a reflex edge).

Next, we describe how to maintain $\mathcal{F}(S, \Pi)$ as the points in S move. For the time being, we assume that Π is some triangulation of a subset $R \subseteq S$ and that motion is such that the topology of Π does not change. That is, there is no insertion or deletion of edges in Π. In addition to ordering and visibility events, a new type of event, called *crossing events*, arises when a point of $S \setminus R$ crosses an edge of Π. In the following, we discuss each of them.

Ordering event. There are two points $p_i, p_j \in S$ so that (i) $p_i(t_0)$ is visible from $p_j(t_0)$ with respect to Π, and (ii) $y(p_i(t_0)) = y(p_j(t_0))$. If $p_i(t_0)$ is not visible from $p_j(t_0)$, then $\mathcal{F}(S, \Pi)$ does not change at t_0. Note that, the visibility relation between two points can change only by a crossing or a visibility event described below.

At each ordering event, $\mathcal{F}(S, \Pi)$ is updated in exactly the same way as in Section 2.

Visibility event. $\rho(p_i(t_0))$ or $\gamma(p_i(t_0))$ changes for some p_i, and p_i does not cross any edge of Π during $[t_0^-, t_0^+]$. We process this event in the same way as in Section 2. Note that $\rho(p_i(t_0))$ could also change as p_i crosses some edge from Π, which will be covered by the crossing event below.

Crossing event. A point p_i crosses an edge $p_j p_k$ of Π at time t_0; assume $y(p_k(t_0)) > y(p_j(t_0))$. Recall that Π is a triangulation of a subset of S. Π partitions $\mathbb{R}^2$ into several connected components, each region being either a triangle of Π or the exterior of conv(R). Suppose p_i moves from the region Δ^- to Δ^+. Let S_{Δ^-} (resp. S_{Δ^+}) be the subset of points of S in Δ^- (resp. Δ^+) at t_0^-. Then the crossing event corresponds to deleting p_i from S_{Δ^-}, inserting it into S_{Δ^+}, and updating the fan triangulation in Δ^- and Δ^+. First, let us consider Δ^-.

Given an arbitrary point $q \in S$, the *star* of q, denoted by St(q), is the union of triangles adjacent to q; St(q) is a star-shaped polygon with q in its kernel — every point in St(q) is visible from q. The *link* of q, denoted by Lk(q), is defined as ∂St(q). Lk(q) is a closed polygonal curve. If $q \in \partial\,\text{conv}(S)$, then $q \in \text{Lk}(q)$; otherwise q lies in the interior of St(q). Given the fan triangulation of S, Lk(q) consists of a convex chain, corresponding to the up neighbors of q, a y-monotone polygonal chain, and one more edge that connects the lowest neighbor of p to an up neighbor of q (or to q if $q \in \partial\,\text{conv}(S)$)); see Figure 4.

If $p_i \in \partial\,\text{conv}(S)$ at time t_0^- (in which case Δ^- represents the exterior of conv(R)), we can simply remove edges $p_i p_k$ and $p_i p_j$ from the triangulation. We now assume that p_i does not lie on conv(S) at time t_0^-, implying that $p_i \notin \text{Lk}(p_i)$. The following lemma is straightforward to prove.

LEMMA 3.1. *Within Δ^-, any edge from $\mathcal{F}(S(t_0^-), \Pi)$ not incident upon p_i at time t_0^- is present in $\mathcal{F}(S(t), \Pi)$ for all $t \in [t_0^-, t_0^+]$.*

In view of Lemma 3.1, we delete the edges incident upon p_i at time t_0^- and re-triangulate within $\text{Lk}(p_i(t_0^-))$. The portion of the triangulation on and outside $\text{Lk}(p_i(t_0^-))$ remains unchanged. We re-triangulate within $\text{Lk}(p_i(t_0^-))$ as follows (Figure 4). Let $Q = \langle p_k = q_0, q_1, \cdots, q_w, q_{w+1}, \cdots, q_u = p_j \rangle$ be the sequence of vertices in Lk(p_i), where $q_0, q_1, \cdots, q_w$ is the convex chain formed by the up neighbors of p_i, and $y(q_{w+1}) > y(q_{w+2}) > \cdots > y(q_u)$. Since $q_0, q_1, \cdots, q_w$ are already in convex position, we only visit vertices q_{w+1} to q_u in order. Suppose we have processed $q_{w+1}, \cdots, q_{z-1}$, i.e., added the new edges incident upon them. We maintain the subsequence of Q

$$\Phi^{(z-1)} = \langle q_{z-1} = \rho^0(q_{z-1}), \rho(q_{z-1}), \rho^2(q_{z-1}), \rho^3(q_{z-1}), \cdots \rangle,$$

i.e., the vertices that appear on the left boundary of the convex hull of $q_0, \cdots, q_{z-1}$. If $q_{z-1}, \rho(q_{z-1}), \cdots, \rho^l(q_{z-1})$ are visible from q_z, we then add the edges $q_z q_{z-1}, q_z \rho(q_{z-1}), \cdots,$

$q_z\rho^l(q_{z-1})$, delete $q_{z-1}, \cdots, \rho^{l-1}(q_{z-1})$ from the sequence, and set $\Phi^{(z)} = \langle q_z, \rho(q_z) = \rho^l(q_{z-1}), \rho^{l+1}(q_{z-1}), \cdots \rangle$.

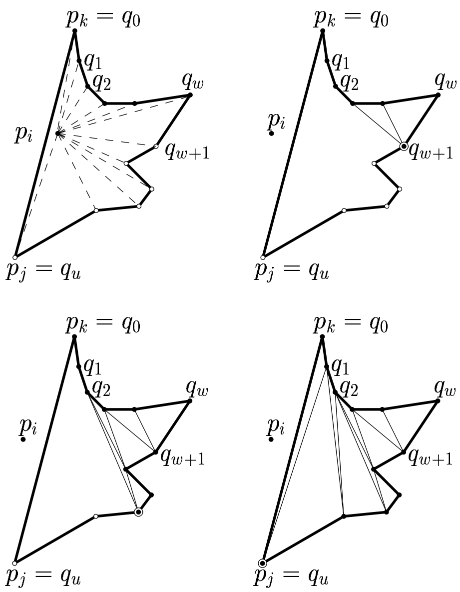

Figure 4: Re-triangulation of $\mathrm{St}(p_i(t_0^-))$ **after deleting** p_i**. The thick polygonal chain is** $\mathrm{Lk}(p_i(t_0^-))$**.**

Next, we describe how to insert p_i into Δ^+ and construct $\mathcal{F}(S_{\Delta^+} \cup \{p_i\})$ at time t_0^+ from $\mathcal{F}(S_{\Delta^+})$. Roughly speaking, we need to do opposite of what we did in Δ^-. That is, we identify $\mathrm{Lk}(p_i(t_0^+))$ in $\mathcal{F}(S_{\Delta^+} \cup \{p_i\})$, delete the edges of $\mathcal{F}(S_{\Delta^+})$ that lie within the polygon formed by $\mathrm{Lk}(p_i(t_0^+))$, and connect p_i to all the vertices in $\mathrm{Lk}(p_i(t_0^+))$ to form $\mathrm{St}(p_i(t_0^+))$. The following procedure performs these steps simultaneously. Let p_l be the vertex in Δ^+ adjacent to the edge p_jp_k at time t_0^-. If p_l does not exist, then p_jp_k is an edge of $\mathrm{conv}(S)$, and p_i becomes a vertex of $\mathrm{conv}(S)$, in which case we simply add the triangle $p_ip_jp_k$. We thus assume that p_l exists. We add the edges p_ip_j, p_ip_k, and p_ip_l. We maintain a stack $\mathcal{S}$ of triangles. Initially, we push $p_ip_lp_j$ and $p_ip_kp_l$ to $\mathcal{S}$ (with the latter being on the top of $\mathcal{S}$). We perform the following procedure until $\mathcal{S}$ becomes empty. An example is illustrated in Figure 6. For a triangle $p_ip_wp_z$ with $y(p_w) > y(p_z)$, we define the region $\tau(p_ip_wp_z)$ to be the intersection of the following three halfplanes: (i) $y > y(p_z)$; (ii) the halfplane bounded by the line p_ip_w containing p_z; and (iii) the halfplane bounded by the line p_wp_z that does not contain p_i; see Figure 5. Intuitively, this region contains points that potentially should be visible to p_i if edge p_wp_z does not exist.

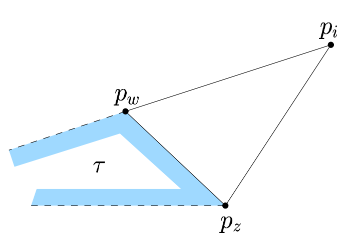

Figure 5: The region τ **bounded by three halfplanes.**

(1) Remove the top triangle $p_ip_wp_z$ from $\mathcal{S}$. Assume that $y(p_w) > y(p_z)$.

(2) If p_wp_z is an edge of the convex hull, or $y(p_z) > y(p_i)$, go to Step 1.

(3) Let p_v be the other vertex adjacent to edge p_wp_z. If $p_v \notin \tau(p_ip_wp_z)$, go to Step 1.

(4) Delete the edge p_wp_z, and insert the edge p_ip_v (edge flip).

(5) Push the triangle $p_ip_vp_z$ to $\mathcal{S}$.

(6) Push the triangle $p_ip_wp_v$ to $\mathcal{S}$.

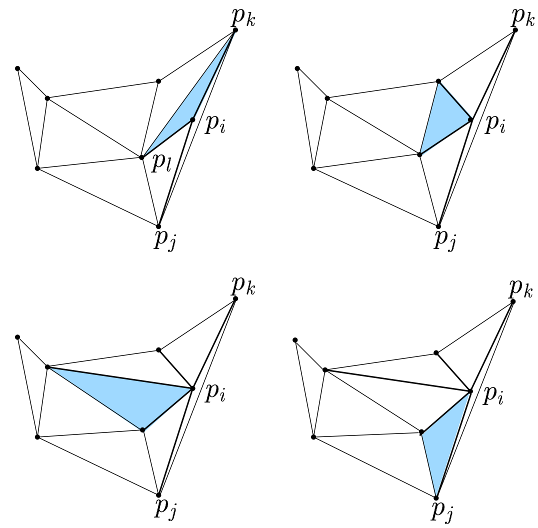

Figure 6: Updating $\mathcal{F}(S_{\Delta^+})$ **after inserting** p_i **— the shaded triangle is being processed.**

The correctness of the above procedure is stated in the next lemma. For notational convenience, let F denote the triangulation constructed by above procedure, and F^- and F^+ the fan triangulation within Δ^+ at time t_0^- and t_0^+ respectively. Observe that if an edge from F or F^+ does not have p_i as an endpoint, then the edge is present in F^- as well. Furthermore, it follows from Lemma 3.1 that if any edge incident upon a vertex q is present in F^-, but not in F^+ (or F), then q is incident upon p_i in F^+.

Lemma 3.2. *F as constructed by the above procedure is the same as F^+.*

Proof. Let $Q = \langle p_k = q_0, q_1, \cdots, q_u = p_j \rangle$ be the sequence of neighbors of p_i in the resulting triangulation F. Again, assume $q_0, \cdots, q_w$ is the convex chain formed by the up neighbors of p_i in F. To prove the lemma, we need to show that (i) any p_iq_z for $0 \le z \le u$ exists in F^+ and (ii) for any $p_ip_z \in F^+$, $p_z \in Q$. The first claim can be shown by induction on the order in which the edges were added by the above procedure. We sketch the proof for the second claim below.

The second claim is proven by contradiction. In particular, let q be the point with smallest y-coordinate such that $q \notin Q$

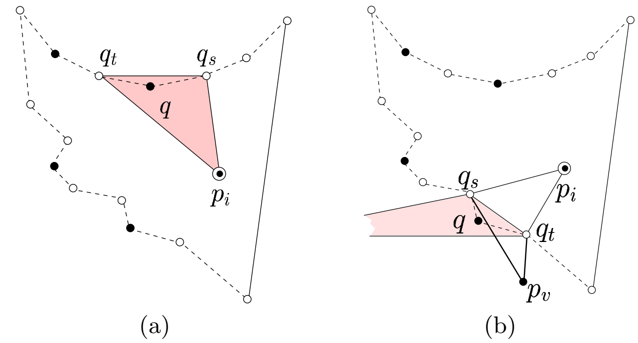

Figure 7: Dotted polygon is $\mathrm{Lk}(p_i)$, with empty dots being vertices from Q. (a) q lies on the convex chain – it lies inside triangle $p_iq_sq_t$; and (b) q lies in the valid region (shadowed) τ, but triangle $p_vq_sq_t$ prevents vertex q being processed.

and $p_iq \in F^+$. Let q_s and q_t be the two consecutive vertices from Q so that q lies inside the wedge formed by p_iq_s and p_iq_t; $t = (s+1) \mod (u+1)$. We now distinguish two cases, either of which will lead to a contradiction.

1. $0 \le s < t \le w$. As $q_0, \cdots, q_w$ form a subset of the up neighbors of p_i in F^+, triangle $p_iq_sq_t$ contains point q, which is not possible (Figure 7 (a)). Contradiction.
2. Otherwise. Suppose $y(q_s) \ge y(q_t)$. Since p_i is visible from q in F^+, it is easy to verify that q lies in region τ as described earlier in the procedure (see Figure 5 with $p_w = q_s$ and $p_z = q_t$, and see Figure 7 (b)). On the other hand, as q_s and q_t are two consecutive neighbors of p_i in F, triangle $p_iq_sq_t$ must have been processed at some moment. The fact that p_iq was not added at that time implies that there was another triangle, say, $p_vq_sq_t$, is incident upon edge q_sq_t (Figure 7 (c)). Because p_iq is visible in F^+At least one of the edges p_vq_s and p_vq_t should not exist in F^+. Therefore p_v is incident upon p_i in F^+. If $p_v \notin Q$, then by our assumption, $y(p_v) < y(q)$. This leads to contradiction as the triangle $p_vq_sq_t$ then cannot have existed in F^- in this case. Otherwise, if $p_v \in Q$, then $y(p_v) > y(q_s)$ or $y(p_v) < y(q_t)$. In either case, it can be shown that p_i is contained in triangle $p_vq_sq_t$. Contradiction.

This proves the second claim, and the lemma follows. □

Note that the number of topological changes to the triangulation, thus the time spent in processing a crossing event, are proportional to the old degree plus the new degree of p_i. We will use this fact later in Section 5.

To detect all the three types of events, we can maintain the same three families of certificates as in the case for fan triangulation.

4. HIERARCHICAL FAN TRIANGULATION

We now use the constrained fan triangulation to define a hierarchical fan triangulation $\mathcal{F}$. Let $\emptyset = R_0 \subseteq R_1 \subseteq \cdots \subseteq$

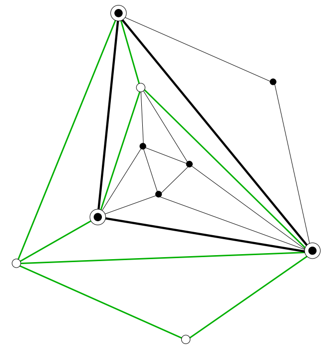

Figure 8: A hierarchical fan triangulation with three levels — points in the first level are denoted by double circles, second by empty dots, and third by solid dots.

$R_w = S$. Set $\mathcal{F}_0 = \emptyset$, and for $i \ge 1$, define $\mathcal{F}_i = \mathcal{F}(R_i, \mathcal{F}_{i-1})$, i.e. $\mathcal{F}_i$ is the constrained fan triangulation of R_i with respect to $\mathcal{F}_{i-1}$. By construction, $\mathcal{F}_1$ is the fan triangulation of R_1 and $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$. Set $\mathcal{F} = \mathcal{F}_w$. See Figure 8 for an example.

Hereditary event. As the points move, each $\mathcal{F}_i$ deforms continuously. The topology of $\mathcal{F}$ changes when there is an ordering, visibility, or crossing event in one of $\mathcal{F}_i$'s. Furthermore a topology change in $\mathcal{F}_i$ propagates changes in $\mathcal{F}_j$ for $j > i$, since the insertion or deletion of an edge in $\mathcal{F}_i$ affects the visibility of points in R_j. We refer to an event in $\mathcal{F}_j$ caused by another event in $\mathcal{F}_i$, for $i < j$, as a *hereditary event* of $\mathcal{F}_j$. The ordering, visibility, and crossing events can be processed as described in Sections 2 and 3. We process hereditary events as follows. If a triangle Δ in $\mathcal{F}_i$ is destroyed due to an edge insertion or deletion, we simply delete all the edges of $\mathcal{F}$ lying inside Δ. As destroying a triangle unavoidably leads to creation of other triangle(s), we only describe how we perform reconstruction after creating a new triangle. Suppose a triangle Δ is created in $\mathcal{F}_i$, we reconstruct affected portion of $\mathcal{F}_j$, for $j > i$. More specifically, let $R_{i+1}^{\Delta} = R_{i+1} \cap \Delta$. We construct the fan triangulation $\mathcal{F}(R_{i+1}^{\Delta})$, and recursively construct the hierarchical fan triangulation of $S \cap \tau$ inside each triangle τ of $\mathcal{F}(R_{i+1}^{\Delta})$. The time spent in updating $\mathcal{F}(R_{i+1}^{\Delta})$ for each newly created triangle Δ can be bounded by $O(|R_{i+1}^{\Delta}|)$.

In the next section, we analyze the performance of the hierarchical fan triangulation, assuming that R_i is a random subset of R_{i+1} of an appropriate size.

5. ANALYSIS

In the hierarchical fan triangulation $\mathcal{F}$ defined in Section 4, let $w = \left\lceil \sqrt{\log n / \log \log n} \right\rceil$ be the number of levels in $\mathcal{F}$, and R_{i-1} be a random subset of R_i of size

$$\min \left\{ |R_i|, 5|R_i|^{1-1/i} \log |R_i| \right\}, \tag{1}$$

for $1 < i \le w$. In this section, we show that $\mathcal{F}$ has near-quadratic complexity as promised. To do this, let us first focus on a specific level of the construction of $\mathcal{F}$. For notational convenience, let $P = R_i$, $R = R_{i-1}$, $\mathcal{T}(P) = \mathcal{F}_i$, $\mathcal{T}(R) = \mathcal{F}_{i-1}$, $n = |P|$ and $r = n^{1-1/i}$, for some $1 < i \le w$. It follows that $\mathcal{T}(P) = \mathcal{F}(P, \mathcal{T}(R))$, and $|R| =$

$\min\{n, 5r \log n\}$.

Let $p_1, p_2, p_3 \in P$, $t, t_1, t_2 \in \mathbb{R}$, $m \in \mathbb{N}$ and $h(p_1, p_2)$ be the open halfspace to the left of the oriented line $\overrightarrow{p_1p_2}$. We define

$$\begin{aligned}
\langle p_1, p_2, p_3; t\rangle &= \{p \in P \mid p(t) \text{ lies inside the triangle } p_1(t)p_2(t)p_3(t)\},\\
\langle p_1, p_2; t_1, t_2\rangle_s &= \{p \in P \mid p(t) \text{ lies upon the segment } p_1(t)p_2(t) \text{ for some } t \in [t_1, t_2]\},\\
\langle p_1, p_2; t_1, t_2\rangle_h &= \{p \in P \mid p(t) \in h(p_1(t), p_2(t)) \text{ for some } t \in [t_1, t_2]\},\\
\langle p_1; m; t\rangle_\rho &= \{\rho^k(p_1(t)) \mid 1 \le k \le m\},\\
\langle p_1; m; t\rangle_\gamma &= \{\gamma^k(p_1(t)) \mid 1 \le k \le m\}.
\end{aligned}$$

We then let

$$\begin{aligned}
\mathcal{R}_1 &= \{\langle p_1, p_2, p_3; t\rangle \mid p_1, p_2, p_3 \in P, t \in \mathbb{R}\},\\
\mathcal{R}_2 &= \{\langle p_1, p_2; t_1, t_2\rangle_s \mid p_1, p_2 \in P, t_1, t_2 \in \mathbb{R}\},\\
\mathcal{R}_3 &= \{\langle p_1, p_2; t_1, t_2\rangle_h \mid p_1, p_2 \in P, t_1, t_2 \in \mathbb{R}\},\\
\mathcal{R}_4 &= \{\langle p; m; t\rangle_\rho \mid p \in P, m \in \mathbb{N}, t \in \mathbb{R}\},\\
\mathcal{R}_5 &= \{\langle p; m; t\rangle_\gamma \mid p \in P, m \in \mathbb{N}, t \in \mathbb{R}\}.
\end{aligned}$$

Finally, let $\mathbb{X}(P)$ be the range space $(P, \mathcal{R})$ with $\mathcal{R} = \bigcup_{1 \le j \le 5} \mathcal{R}_j$. We have the following lemma.

LEMMA 5.1. $|\mathcal{R}| = O(n^5)$.

PROOF. Consider three points p_1, p_2 and p_3 of P. As t increases, $\langle p_1, p_2, p_3; t\rangle$ changes only when some point $q \in P$ moves in or out of the triangle $p_1p_2p_3$. For any point q, this can only happen constant number of times. Thus $\langle p_1, p_2, p_3; t\rangle$ can change $O(n)$ times as t goes from $-\infty$ to $+\infty$ for fixed p_1, p_2, and p_3. There are $O(n^3)$ different choices of p_1, p_2, p_3, implying that $|\mathcal{R}_1| = O(n^4)$. Similarly we can prove $|\mathcal{R}_2| = O(n^4)$ and $|\mathcal{R}_3| = O(n^4)$. For any $p, q \in P$, let $\theta(p, q)$ denote the angle formed by $\overrightarrow{pq}$ and $+x$-direction. The number of changes to $\rho(p)$ is the same as the complexity of the upper envelope of $\{\theta(p, q) \mid q \ne p, q \in P\}$, which can be bounded by $\lambda_s(n)$ [16], where $\lambda_s(\cdot)$ is the maximum length of a Davenport-Schinzel sequence of order s. The value of s depends on the maximum degree of polynomial of the trajectories of points in P; $s = 4$ if points move linearly. For fixed p and m, as t increases, $\langle p; m; t\rangle_\rho$ changes only when $\rho(q)$ changes for some $q \in P$. There are at most n different choices for p and m each, thus we have $|\mathcal{R}_4| = O(n^3\lambda_s(n))$, which is slightly larger than $O(n^4)$. Similarly $|\mathcal{R}_5| = O(n^3\lambda_s(n))$. Putting everything together, we have the lemma. □

As time t goes on, the combinatorial structure of $\mathcal{T}(P)$ may change: features such as edges or triangles may appear or disappear. The *lifetime* of a feature is the period between two of its consecutive appearance and disappearance. If an edge or a triangle occurs more than once, we count each occurrence as a different feature.

By (1), Lemma 5.1, and standard results from random sampling theory [15], R is a $(1/r)$-net of the range space $\mathbb{X}(P)$ with high probability. Thus we obtain the following lemma.

LEMMA 5.2. *With high probability,*

(a) At any time, there are less than n/r points of P inside any triangle of $\mathcal{T}(R)$.

(b) Less than n/r points of P can cross any edge ever appearing in $\mathcal{T}(R)$ during its lifetime.

(c) Less than n/r points of P can ever appear in $h(p, q)$ during the period when $p, q \in R$ are two neighboring points on the convex hull of R in clockwise order.

LEMMA 5.3. *With high probability,*

(a) For any $p \in P, m \in \mathbb{N}, t \in \mathbb{R}$, $|\langle p; m; t\rangle_\rho| \le (|\langle p; m; t\rangle_\rho \cap R| + 1) \cdot n/r$.

(b) For any $p \in P, m \in \mathbb{N}, t \in \mathbb{R}$, $|\langle p; m; t\rangle_\gamma| \le (|\langle p; m; t\rangle_\gamma \cap R| + 1) \cdot n/r$.

PROOF. Suppose that

$$\langle p; m; t\rangle_\rho \cap R = \{\rho^{i_1}(p(t)), \rho^{i_2}(p(t)), \cdots, \rho^{i_h}(p(t))\},$$

where $1 \le i_1 < i_2 < \cdots < i_h \le m$ and $h = |\langle p; m; t\rangle_\rho \cap R|$. Let $i_0 = 0$ and $i_{h+1} = m + 1$. We have

$$\begin{aligned}
\langle p; m; t\rangle_\rho &= \left(\bigcup_{j=1}^{h+1} \langle \rho^{i_{j-1}}(p); i_j - i_{j-1} - 1; t\rangle_\rho\right)\\
&\quad \cup \{\rho^{i_1}(p(t)), \rho^{i_2}(p(t)), \cdots, \rho^{i_h}(p(t))\}.
\end{aligned}$$

Moreover, by properties of the $(1/r)$-net of $\mathbb{X}(P)$, we know that with high probability, $|\langle \rho^{i_{j-1}}(p); i_j - i_{j-1} - 1; t\rangle_\rho| \le n/r - 1$, for $1 \le j \le h + 1$. We thus obtain $|\langle p; m; t\rangle_\rho| \le (h + 1) \cdot (n/r - 1) + h \le (h + 1) \cdot n/r$. (b) can be proved in a similar manner. □

For a set $A \subseteq P$ of points, let $\Psi(A)$ denote the complexity of $\mathcal{T}(A)$. We will bound $\Psi(P)$ in terms of $\Psi(R)$. For simplicity, we will regard $\mathcal{T}(R)$ as a planar subdivision, with the exterior of $\mathrm{conv}(R)$ (i.e., Σ) being one of its faces.

Before we bound the number of events, we prove another lemma. Let $\deg(p(t))$ denote the degree of point $p \in P$ in $\mathcal{T}(P)$ at time t.

LEMMA 5.4. *If a point $p \in P$ lies on an edge of $\mathcal{T}(R)$ at time t_0, then both $\deg(p(t_0^-))$ and $\deg(p(t_0^+))$ are bounded by n/r with high probability.*

PROOF. If $p(t)$ lies in the interior of $\mathrm{conv}(R)$, then we have $\deg(p(t)) \le n/r$ with high probability by Lemma 5.2 (a), because each triangle in $\mathcal{T}(R)$ contains at most n/r points of P. If p crosses an edge p_1p_2 of $\partial\,\mathrm{conv}(R)$ at t_0, say, from inside to outside, then at t_0^+, all points adjacent to p lie in the open halfplane bounded by p_1p_2 that is disjoint from $\mathrm{conv}(R)$. The lemma then follows from Lemma 5.2 (c). A symmetric argument works if p moves from the exterior to the interior of $\mathrm{conv}(R)$ at t_0. □

We now bound the time complexity induced by each type of events as discussed in Section 3 and 4.

Hereditary event. Each hereditary event either causes to re-triangulate $\mathcal{T}(P)$ inside $O(1)$ triangles of $\mathcal{T}(R)$, or inserts or deletes a point that crosses an edge of $\text{conv}(R)$. By Lemma 5.2 (a) and 5.4, we spend $O(n/r)$ time at each event. Since each hereditary event is an event of $\mathcal{T}(R)$, the total time spent in processing hereditary events is $O(n/r) \cdot \Psi(R)$.

Crossing event. By Lemma 5.2 (b), at most n/r points cross an edge of $\mathcal{T}(R)$ during its lifetime. By Lemma 5.4, we spend $O(n/r)$ time at each such event. Since there are $\Psi(R) + O(|R|)$ distinct edges in $\mathcal{T}(R)$ throughout time, the total time spent in processing crossing event is $O(n^2/r^2) \cdot (\Psi(R) + |R|) = O(n^2/r^2) \cdot \Psi(R)$.

Visibility event. There is a change in $\rho(p)$ (or $\gamma(p)$) with respect to $\Delta \in \mathcal{T}(R)$ for some point $p \in P_\Delta$ (recall P_Δ is the set of points of P inside Δ). As discussed earlier, each such event causes $O(1)$ changes to the fan triangulation within Δ, thus to $\mathcal{T}(P)$. Therefore, we only need to bound the number of visibility events.

A visibility event occurs due to the collinearity of p, q and some other point z, where p, q and z lie within the same face, say Δ, of $\mathcal{T}(R)$. If Δ is a triangle in $\mathcal{T}(R)$, then by Lemma 5.2, $O(n/r)$ points can ever appear in Δ during its lifetime. Using a lower-envelope argument as in the proof of Lemma 5.1, it can be shown that the number of visibility events happening to p while $p \in \Delta$ is $O(\lambda_s(n/r))$, where again, $s = 4$ for points moving linearly. Since there are $O(|R| + \Psi(R))$ triangles in $\mathcal{T}(R)$ over entire history, the total number of visibility events inside $\text{conv}(R)$ is $O((n/r) \cdot \lambda_s(n/r) \cdot (|R| + \Psi(R))) = O(n^2 \log n/r^2) \cdot \Psi(R)$.

Next, suppose a point $p \notin \text{conv}(R)$ causes a visibility event. Let l_t (resp. l_b) be the horizontal line passing through the highest (resp. lowest) point of R. We distinguish two cases: (a) p lies between l_t and l_b and to the left (resp. right) of $\text{conv}(R)$, and $\gamma(p)$ (resp. $\rho(p)$) changes; and (b) otherwise.

For case (b), it is easy to verify that it corresponds to the change of $\rho(p)$ or $\gamma(p)$ in P. Therefore, there are $O(n\lambda_s(n)) = O(n^2 \log n)$ number of such events by similar lower-envelope arguments.

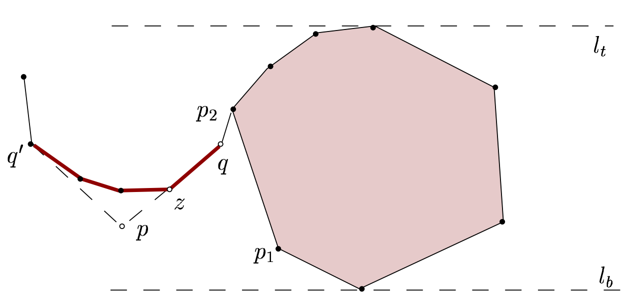

Figure 9: Shadowed region is $\text{conv}(R)$**, and the thick chain is** $\mathcal{V}(p)$**.**

Now consider case (a) (depicted in Figure 9). Assume without loss of generality that p lies to the left of $\text{conv}(R)$. Let p_1p_2 be the edge on $\text{conv}(R)$ that is hit by the ray shooting from p towards its right (assuming p_1, p_2 are in clockwise order). Observe that p, q, z must all be in the triangle pp_1p_2, and therefore $p, q, z \in h(p_1, p_2)$. Thus, during the period when p_1, p_2 are two neighboring points on $\text{conv}(R)$ in clockwise order, the number of events that three points of P in $h(p_1, p_2)$ become collinear is bounded by $O(n^3/r^3)$ by Lemma 5.2 (c). Since $\text{conv}(R)$ changes $O(|R|^2)$ times [4], we can have at most $O(n^3|R|^2/r^3)$ such events.

In total, the number of topological changes to the triangulation of P caused by visibility events, and thus the total time spent in processing these events, are bounded by $O(n^3|R|^2/r^3 + n^2 \log n + (n^2 \log n/r^2) \cdot \Psi(R))$.

Ordering event. The time spent at an ordering event at time t_0 when the y-coordinates of two mutually visible points p and q becomes equal is proportional to $|\mathcal{V}_{pq}(t_0)|$, i.e., the number of points that are visible from both p and q at time t_0. If p and q lie inside a triangle face of $\mathcal{T}(R)$, then $|\mathcal{V}_{pq}(t_0)| = O(n/r)$. Since there are $O(n^2)$ ordering events, the total time spent at such events is $O(n^3/r)$. Now assume that p and q do not lie in the interior of $\text{conv}(R)$, and let I be the set of such events. Obviously, $|I| \leq n^2$. Denote by m_i the number of points of R on $\mathcal{V}_{pq}(t_0)$ for the i-th such event from I. It follows from Lemma 5.3 that the total time spent in processing this type of events is at most

$$\sum_i (m_i + 2)n/r = O(n^3/r) + (n/r) \cdot \sum_i m_i. \qquad (2)$$

LEMMA 5.5. $\sum_{i \leq |I|} m_i = O(n^2|R|^2/r^2)$.

PROOF. Consider one event from I happening at time t_0 between points p and q. Let $z_1, z, z_2 \in R$ be three consecutive points on $\text{conv}(R)$ in clockwise order at t_0, and $z \in \mathcal{V}_{pq}(t_0)$. Recall that $h(z_1, z)$ is the open halfspace to the left of $\overrightarrow{z_1 z}$. We have that $p, q \in h(z_1, z) \cup h(z, z_2)$ as z is visible to both p and q. For fixed edges $z_1 z$ and zz_2 from $\text{conv}(R)$, z can only be involved in $O(n^2/r^2)$ such events. This is because that by Lemma 5.2 (c), only $O(n/r)$ points ever appear in $h(z_1, z)$ (resp. $h(z, z_2)$) during the lifetime of edge $z_1 z$ (resp. zz_2). Let u_z be the total number of times that z is involved in such an event. As $\text{conv}(R)$ may change $O(|R|^2)$ times as points in R move [4], we have $\sum_{z \in R} u_z = O(n^2|R|^2/r^2)$. Using a simple double counting argument, it is straightforward that $\sum_{i \in I} m_i = \sum_{z \in R} u_z$. This proves the lemma. $\square$

It follows from the above lemma that the total time spent in processing ordering events can be bounded by $O(n^3/r + n^3|R|^2/r^3) = O(n^3|R|^2/r^3)$.

Putting everything together, we obtain the following recurrence for $\Psi(P)$.

$$\Psi(P) \leq O(n^{2+1/i} \cdot \log^2 n) + O(n^{2/i} \cdot \log n) \cdot \Psi(R).$$

Hence, coming back to the hierarchical fan triangulation, let n_i be the size of R_i,

$$\Psi(n_i) \leq O\left(n_i^{2/i} \log n_i\right) \cdot \Psi(n_{i-1}) + O\left(n_i^{2+1/i} \log^2 n_i\right),$$

where $n_{i-1} = |R_{i-1}| = \min\{n_i, 5n_i^{1-1/i} \log n_i\}$. The solution to the above recurrence is $\Psi(n) = n^2 2^{O(\sqrt{\log n \cdot \log\log n})}$. We conclude with the following main result. The proof for the more general case when only k points are moving is omitted from this extended abstract.

THEOREM 5.6. *The complexity of the hierarchical fan triangulation $\mathcal{F}$ of a set S of n linearly moving points in the plane is $O(n^2 2^{O(\sqrt{\log n \cdot \log\log n})})$.*

6. EXTENSION

As a special case of our problem, let us consider a scenario in which only k out of n points are moving. By extending our previous technique, we can show that there exists a triangulation whose number of topological changes is roughly $O(nk)$. We describe this triangulation briefly in this section.

The overall framework follows that for the case with n moving points. The major difference is that, instead of sampling the points all at once at each level of the hierarchical fan triangulation, we sample the static and moving points separately. For any point set P, let $P^a \subseteq P$ be the set of static points in P, and $P^b \subseteq P$ be the set of moving points in P.

Set $w = \left\lceil \sqrt{\log k / \log\log n} \right\rceil$. Let $\emptyset = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_w = S$, such that for $1 < i \leq w$, R_{i-1}^a is a random subset of R_i^a of size

$$\min\left\{|R_i^a|, O(|R_i| \cdot |R_i^b|^{-1/i} \log |R_i^a|)\right\},$$

and R_{i-1}^b is a random subset of R_i^b of size

$$\min\left\{|R_i^b|, O(|R_i^b|^{1-1/i} \log |R_i^b|)\right\}.$$

We then construct the hierarchical fan triangulation $\mathcal{F}$ of S as described in Section 4.

Let $n_i = |R_i|$ and $k_i = |R_i^b|$. By the same method as in [2], one can show that the fan triangulation $\mathcal{T}(R_1)$ changes $O(k_1^{4/3}\lambda_d(n_1)) = O(n_1 k_1^{4/3} \log n_1)$ times. Using this fact, and by similar analysis as in the preceding section, it can be shown that the complexity of $\mathcal{T}(R_i)$ is bounded by $n_i k_i^{1+1/i} \cdot \log^{4i} n_i \cdot 2^{ci}$ for some constant c, for $1 \leq i \leq w$. Without going into the detail, we conclude with the following theorem.

THEOREM 6.1. *For $0 \leq k \leq n$, there exists a triangulation with complexity $O(nk \cdot 2^{O(\sqrt{\log k \cdot \log\log n})})$ for a set of $(n-k)$ static and k linearly moving points in the plane.*

We remark that, with a few minor changes, our proof also works for moving points with algebraic trajectories of bounded degree.

7. CONCLUSIONS

In this paper, we have described a triangulation whose number of topological changes for a set of n linearly moving points in $\mathbb{R}^2$ is bounded by $O(n^2 2^{O(\sqrt{\log n \cdot \log\log n})})$. The time needed for maintaining this triangulation using KDS is of roughly the same order. Our result almost matches the $\Omega(n^2)$ lower bound, and improves over the previously best known result [2] by nearly a factor of $n^{1/3}$. If only k points of the point set are moving, the complexity can be reduced to $O(nk \cdot 2^{O(\sqrt{\log k \cdot \log\log n})})$. These results also hold for moving points with algebraic trajectories of bounded degree.

Although the triangulation that we construct is conceptually simple, it is not trivial to compute it deterministically and efficiently using standard derandomization techniques, as it requires computing the ε-net for a fairly large range space. It will be interesting to find a simple triangulation, easy to implement, and with near-quadratic complexity. In particular, it remains a most intriguing open problem to find out whether the popular Euclidean Delaunay triangulation has near-quadratic complexity. We leave these problems for further research.

Another interesting problem is whether Steiner points can help to reduce the complexity of the triangulation. In general, if linear number of Steiner points are allowed, the answer is negative: Agarwal *et al.* [1] showed that the $\Omega(n^2)$ lower bound still holds with $O(n)$ Steiner points and these Steiner points can move along any continuous trajectories. This bound is tight even when we require that all the Steiner points should lie inside the convex hull of the moving points at any moment. We describe such a triangulation briefly. First construct the convex hull of S, and connect each point to the next point in increasing x-order. In the resulting subdivision, compute a trapezoidal decomposition and the set of Steiner points added are the set of intersections between the vertical lines (coming from the trapezoidal decomposition) and the convex hull. The final triangulation is obtained by refining each trapezoid into two triangles. However, the trajectories of the Steiner points constructed this way do not have constant description (worst case the descriptive complexity of a Steiner point can be $\Omega(n)$). It is desirable to maintain a simple and practical triangulation with $O(n^2)$ complexity while adding only $O(n)$ Steiner points, with each has a descriptive complexity of $O(1)$.

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