Lecture 6: Computation of Simplicial Homology: Matrix view

Topics in Computational Topology: An Algorithmic View

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1 Boundary matrix

In order to calculate the Betti numbers in an algorithmic way, we must develop an formal method. The boundary matrix provides us such an approach.

Given a simplicial complex K, we have its pth chain group, C_p, which is the group of all possible formal sums of p-simplices in K. More strictly speak, $\mathsf{C}_p = \mathbb{Z}_2^{n_p}$, where n_p is the number of p-simplices in K and $\mathbb{Z}_2$ is the cyclic group $\mathbb{Z}/2\mathbb{Z}$. With the pth boundary operator $\partial_p : \mathsf{C}_p \to \mathsf{C}_{p-1}$ we defined previously, we have $\mathsf{Z}_p = Ker(\partial_p)$ and $\mathsf{B}_{p-1} = Im(\partial_p)$. Similarly to the abelian group representation of C_p, we also have that $\mathsf{Z}_p = \mathbb{Z}_2^{z_p}$ and $\mathsf{B}_p = \mathbb{Z}_2^{b_p}$, where z_p, b_p is called the rank of Z_p and B_p respectively. By the Rank-nullity theorem in linear algebra[3], we have

$$\mathsf{C}_p \simeq \mathsf{Z}_p \oplus \mathsf{B}_{p-1} \text{ or } \mathbb{Z}_2^{n_p} \simeq \mathbb{Z}_2^{z_p} \oplus \mathbb{Z}_2^{b_{p-1}}$$

which indicates the important result

$$n_p = z_p + b_{p-1} \tag{1}$$

Since we also know that pth Betti number $\beta_p = z_p - b_p$, we can calculate the β_p for each p as long as we can decompose n_p as (1).

Now let us give the way to decompose n_p with the boundary matrix. Let $\mathsf{C}_p = \{\alpha_1, \alpha_2, \ldots, \alpha_{n_p}\}$ and $\mathsf{C}_{p-1} = \{\tau_1, \tau_2, \ldots, \tau_{n_{p-1}}\}$. Define

$$M_p = \begin{array}{c} \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{n_{p-1}} \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots & \alpha_{n_p} \end{array} \\ \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-1}}^2 & \cdots & a_{n_{p-1}}^{n_p} \end{pmatrix} \end{array}$$

where $a_i^j = 1$ if and only if ith $p-1$ simplex τ_i is a face of the jth p simplex α_j, otherwise, $a_i^j = 0$. Then we have

$$\partial_p \alpha_j = \sum_{i=1}^{n_{p-1}} a_i^j \tau_i$$

From an algebraic point of view, if we use $\{\tau_1, \tau_2, \ldots, \tau_{n_{p-1}}\}$ as the basis of B_{p-1}, the jth column of M_p is just the coordinate of $\partial_p \alpha_j$ under that basis. Thus, the pth boundary matrix actually encodes all the possible relationship between pth simplices and their boundaries.

Given a pth chain c in K, we have $c = \sum_{j=1}^{n_p} c_j \alpha_j$ and its boundary

$$\partial_p c = \partial_p \sum_{j=1}^{n_p} c_j \alpha_j = \sum_{j=1}^{n_p} c_j \partial_p \alpha_j = \sum_{j=1}^{n_p} c_j \sum_{i=1}^{n_{p-1}} a_i^j \tau_i = \sum_{i=1}^{n_{p-1}} (\sum_{j=1}^{n_p} a_i^j c_j) \tau_i$$

Thus, under the basis of $\{\tau_1, \tau_2, \ldots, \tau_{n_{p-1}}\}$, the coordinate of $\partial_p c$ is

$$\begin{bmatrix} a_1^1 & a_1^2 & \ldots & a_1^{n_p} \\ a_2^1 & a_2^2 & \ldots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-1}}^2 & \ldots & a_{n_{p-1}}^{n_p} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_p} \end{bmatrix}$$

Since it is true for any p chain in K, we have $\mathsf{B}_{p-1} = Im(\partial_p) = SPAN\{col_1, col_2, \ldots, col_{n_p}\}$, where col_j is the jth column of boundary matrix M_p. With the knowledge of linear algebra, we have $b_{p-1} = Dim(\mathsf{B}_{p-1}) = rank(M_p)$. Now we can see that we have related b_{p-1} to rank of the boundary matrix, which can be calculated through the well-known linear algebra technique, Gaussian elimination.

Before we move forward, the following example will help to illustrate the concept we discuss so far.

Example 1.1 *Consider the the 2-dimensional faces of the tetrahedron with four points a, b, c, d and the boundary operator ∂_2. Then we have*

$$M_2 = \begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array} \begin{array}{c} \begin{array}{cccc} abc & abd & acd & bcd \end{array} \\ \begin{pmatrix} 1 & 1 & & \\ 1 & & 1 & \\ & 1 & 1 & \\ 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix} \end{array}$$

For example, the 3 simplex abc has 3 faces, ab, ac, bc. Thus the column of abc has 1s on the corresponding entries. And the 2 chain, $abc + abd$, has faces, ac, ad, bc, bd, and the sum of the columns corresponding to abc and abd has 1s on its four faces.

2 Smith Normal Form

As we discussed in the previous section, we can obtain the Betti numbers β_p for any p as long as we can find the rank for M_p for each p. Now let us introduce an algorithmic method to decompose n_p as (1).

First of all, let us introduce two basic operations of columns on the boundary matrix, which don't change the rank of the matrix.

1. Exchanging two column, say j_1th and j_2th column, let $\hat{a}_i^{j_1} = a_i^{j_2}$ and $\hat{a}_i^{j_2} = a_i^{j_1}$ for $i = 1, 2, \ldots, n_{p-1}$, where $\hat{a}_i^j$ is the entry of new matrix.

2. Adding one column to another, let $\hat{a}_i^{j_1} = a_i^{j_1} + a_i^{j_2}$.

And similar operations can be define on the rows of boundary matrix. Moreover, all these operations can be view as multiplying M_p by one elementary matrix[2]. And we know that these operations on boundary matrix won't change its rank. Using the two basic operations we mentioned before, we can reduce the boundary matrix M_p into the following form, or **Smith Normal Form** with Gaussian elimination.

$$S_p = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & 0 & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & 0 & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

After reducing the boundary matrix into this form, we can easily find $rank(M_p) = b_{p-1}$ by find the number of 1s on the diagonal of the S_p. Now let us describe the process of Gaussian elimination, which is basic in linear algebra.

Here, we just use the process introduced in [1]. Let $T = M_p$ initially. Using at most two exchange operations, we can move a 1 to the upper left position of T. And with at most $n_p - 1$ column and $n_{p-1} - 1$ row additions, we can cancel out all the 1s in the first row and first column. Then let T be its submatrix obtained by removing the first row and first column and do the same thing recursively. Let x be the row and column number of upper left element of the submatrix we consider. Initially, $x = 1$.

Algorithm 1 Reduce(x)

if $\exists k \geq x, l \geq x$ with $a_k^l = 1$ **then**
  exchange rows x and k; exchange columns x and l;
  for $i = x + 1$ to n_{p-1} **do**
    if $a_i^x = 1$ **then**
      add row x to row i
    end if
  end for
  for $i = x + 1$ to n_p **do**
    if $a_x^j = 1$ **then**
      add column x to column j
    end if
  end for
  $Reduce(x + 1)$
end if

With this procedure, we have at most n_{p-1}^2 row operation and at most n_p^2 column operations and $O(n_{p-1} n_p (n_p + n_{p+1}))$ for total complexity.

3 Basis of Boundary group B_{p-1} and cycle group Z_p

Besides the interest on Betti number, if we are also interested in the basis of the Boundary group B_{p-1} and cycle group Z_p. Gaussian elimination will help us as long as we also record the change on the representative p chains for the columns. In particular, we associate with the j-th column also a p-chain γ_j. Initially, $\gamma_j = \alpha_j$; note that at this point, $col_j = \partial \gamma_j$. We now reduce the matrix so that each remaining columns are linearly independent using only the following operations.

1. For exchange operation of columns, we also exchange the representatives of columns.

$$\hat{\gamma}_{j_1} \leftarrow \gamma_{j_2}, \hat{\gamma}_{j_2} \leftarrow \gamma_{j_1}$$

2. For addition of j_1 column to j_2 column

$$\hat{\gamma}_{j_2} \leftarrow \gamma_{j_1} + \gamma_{j_2}$$

Let γ_j^k and col_j^k denote the j-th column and its associated p-chain after k operations. Note that none of the two operations above change the following fact, which means that we maintain the following invariance during the reduction:

$$\textbf{Invariance } col_j^k = \partial \gamma_j^k.$$

In the end, after the initial boundary matrix M_p is reduced to R_p where all columns are linearly independent, it is easy to see that the remaining non-zero columns form a basis of B_{p-1}.

To obtain a basis for Z_p, collect the set of p-chains $\Pi = \{\gamma_i \mid col_i \text{ is all zero } \}$. Obviously, each p-chain in Π is a p-cycle, as its corresponding column (which is its boundary due to the invariance that we maintain) is empty. There are exactly z_p number of cycles in Π. Furthermore, it is easy to verify that all these p-chains are independent. This is because at the beginning, all γ_i^0 are independent. The column operations we perform do not change ths fact. Hence the set of γ_i^k at any stage k are always independent. It then follows that p-cycles in Π form a basis for the p-th cycle group Z_p.

Now let us use these two operations to compute the basis of boundary for the example (1.1).

Example 3.1 *As we can see in the matrix in example (1.1), in order to cancel the* 1*s in first row, we need to add* 1*st column to* 2*nd column. After that, we got the following matrix*

$$\begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array}
\begin{array}{c} \begin{array}{cccc} abc & \substack{abd \\ abc} & acd & bcd \end{array} \\
\begin{pmatrix} 1 & & & \\ 1 & 1 & 1 & \\ & 1 & 1 & \\ 1 & 1 & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix} \end{array}$$

We continue this way to eleminate, for each diagonal 1*, all entries to its right in its corresponding column. In the end, we obtain the following reduced matrix* R_2*.*

$$\begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array}
\begin{array}{c} \begin{array}{cccc} abc & \substack{abd \\ abc} & \substack{abd \\ abc \\ acd} & \substack{abd \\ abc \\ acd \\ bcd} \end{array} \\
\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & & \\ & 1 & 1 & \\ 1 & 1 & 1 & \\ & & 1 & \end{pmatrix} \end{array}$$

From this reduced matrix R_2*, we can conclude that the columns* $c_1 = abc, c_2 = abd + abc, c_3 = abc + abd + acd$ *for the basis for* B_1 *for tetrahedron. The only column with all* 0*s is the last one, and the associated* p*-chain* $c_4 = abc + abd + acd + bcd$ *is the only non-trivial element in* Z_2 *of tetrahedron.*

References

[1] H. Edelsbrunner and J. Harer. *Computational topology: an introduction*. 2009.

[2] Wikipedia. Elementary matrix. http://en.wikipedia.org/wiki/Elementary_matrix.

[3] Wikipedia. Rank-nullity theorem. http://en.wikipedia.org/wiki/Rank-nullity_theorem.