Lecture 6: Introduction to Simplicial Homology

Topics in Computational Topology: An Algorithmic View

Scribed by: Marc Khoury

1 Convex Set Systems

We begin by exploring several specific simplicial complexes.

1.1 Nerve

Definition 1.1 *Let $\pi = \{S_0, \ldots, S_n\}$ be a finite collection of sets. The* ***nerve*** *consists of all subcollections whose sets have a non-empty common intersection,*

$$Nrv(\pi) = \{X \subseteq \pi | \bigcap X \neq \emptyset\}$$

Notice that the nerve is an abstract simplicial complex since $\bigcap X \neq \emptyset$ and $Y \subseteq X$ implies that $\bigcap Y \neq \emptyset$.

Consider a subcollection $\{S_{i_0}, ..., S_{i_n}\} \subseteq \pi$ of discs with non-empty intersection. Construct the $(n-1)$-simplex σ formed by connecting the centers of each set. The nerve of π is the set of all simplices σ that can be constructed in this manner.

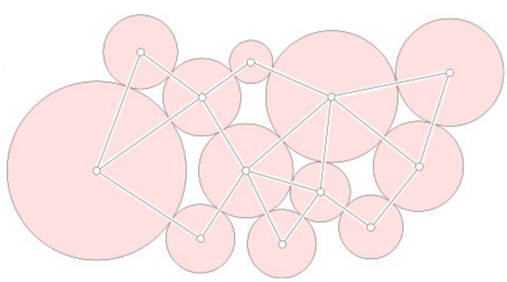

Figure 1: The nerve of a collection of discs. Notice that since no three discs intersect, there is no 2-simplex in the nerve

The nerve of a collection of sets has a geometric realization. Thus we can consider the homotopy type of the nerve. The relationship between the homotopy type of the nerve and the collection of sets is summarized in the *Nerve Theorem*.

Theorem 1.2 *Let π be a collection of closed and convex sets in Euclidean space. Then $Nrv(\pi)$ and the union of the sets in π have the same homotopy type.*

1.2 Čech Complex

Definition 1.3 *Let S be a finite set of points in $\mathbb{R}^d$. For a non-negative radius r we consider the ball $B_x(r) = x + r\mathbb{B}^d$ for each $x \in S$. The Čech complex is defined as*

$$\check{C}ech(r) = \{\sigma \subseteq S | \bigcap_{x \in \sigma} B_x(r) \neq \emptyset\}$$

Notice that the Čech complex is equal to the nerve of the collection of balls. The Čech complex may not have a geometric realization in $\mathbb{R}^d$ because it is likely there exists a high dimension simplex in the complex. As we increase the radius r the balls increase in size and more simplicies are added without losing those that were already present. Thus we have Čech$(r_0) \subseteq$ Čech(r_1) for $r_0 \leq r_1$.

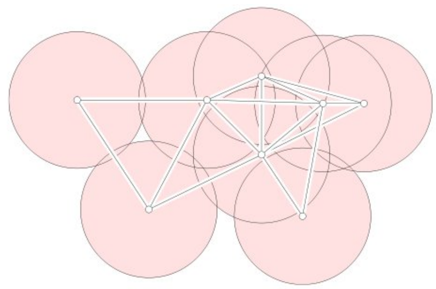

Figure 2: The Čech complex of nine points for radius r. The complex contains nine of the ten possible triangles and two tetrahedra. The only difference between the Čech complex and the Rips complex is the tenth triangle, which belongs only to the Rips complex.

1.3 Rips Complex

Definition 1.4 *The Rips complex of S and r consists of all subsets of diameter at most r,*

$$Rips(r) = \{\sigma \subseteq S | diam(\sigma) \leq 2r\}$$

In the Rips complex, we add any 2 and higher dimensional simplex that we can. For example the tenth triangle in figure 3 would be included in the Rips complex because the three edges were already included. It is easy to see that Č$ech(r) \subseteq Rips(r)$. However we also have the following lemma. A stronger version is available in [1].

Lemma 1.5 *Čech(r) $\subseteq$ Rips(r) $\subseteq$ Čech(2r).*

2 Homology

The fundamental group is a great tool for characterizing holes in topological spaces. Unfortunately it is difficult to work with algorithmically. Homology groups give the best of both worlds. They are useful for characterizing surfaces and provide fast algorithms.

2.1 Chain Complexes

Let K be a simplicial complex. A *p-chain* is a finite formal sum of p-simplices in K, written as $c = \sum a_i \sigma_i$. In this class, we consider the case where the coefficients a_is come from the field $\mathbb{Z}_2$; that is, $a_i = 0$ or 1. Under $\mathbb{Z}_2$ coefficients, a p-chain is simply a collection of p-simplices. We can define a binary operation $+$, over the set of p-chains for a simplicial complex as follows. Let $c_0 + c_1 = \sum a_i \sigma_i + \sum b_i \sigma_i = \sum (a_i + b_i \mod 2) \sigma_i$ — Again, since coefficients are from $\mathbb{Z}_2$, the addition is also addition modulo 2.

Lemma 2.1 *Let K be a simplicial complex and let C_p be the set of p-chains over K. The set C_p with the operator $+$ form a group, denoted $(C_p, +)$. Furthermore $(C_p, +)$ is a free abelian group.*

Proof: The identity element is $\sum 0\sigma_i = 0$. For every chain c we have an inverse $-c = c$ since $c + c = 0$ when the coefficients are chosen from $\mathbb{Z}_2$. Closure under addition is easy to see, since for any two chains we add in this manner we always get another chain. Associativity and commutativity follow from the properties of addition. Lastly the set of simplices $\{\sigma_0, \sigma_1, ..., \sigma_n\}$ from K generate C_p and is the minimal set of such generators. This set forms a basis for C_p. Therefore $(C_p, +)$ is a free abelian group. ∎

Let $[v_0, .., v_p]$ denote a p-simplex. The *boundary* of this p-simplex is the various $(p-1)$-dimensional simplices $[v_0, ..., \hat{v}_i, ...v_p]$, where the ˆ symbol means that v_i has been removed from the sequence $v_0, \ldots, v_p$. We can define the boundary operator ∂_p for a simplex σ as follows. This formulation is courtesy of [2].

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_p]$$

The signs are inserted to take orientation into account, as demonstrated by the following figure.

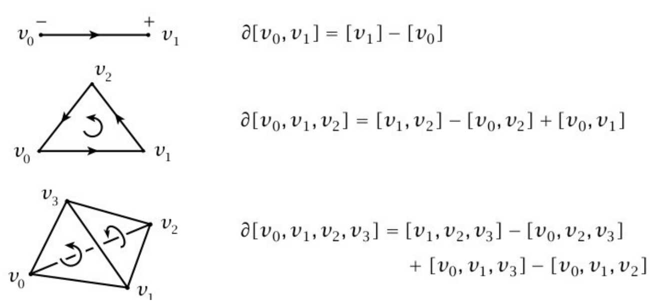

Figure 3: The boundary operator applied to several simplices.

Note that under modulo-2 addition, $a - b = a + b$, and the orientation of simplices does not matter. Hence one can simply write the boundary of a p-simplex is the addition of all its $(p-1)$-faces. For example, $\partial[v_0 v_1] = [v_0] + [v_1]$, etc. The boundary operator has many nice properties. For a given p-chain $c = \sum a_i \sigma_i$, the boundary is the sum of the boundaries of its simplices, $\partial_p c = \sum a_i \partial_p \sigma_i$. Additionally the boundary operator commutes with addtion, $\partial_p(c_0 + c_1) = \partial_p c_0 + \partial_p c_1$. Thus the map $\partial_p : C_p \to C_{p-1}$ is a homomorphism. The *chain complex* is the sequence of chain groups connected by boundary homomorphisms.

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-2}} \ldots$$

2.2 Cycles and Boundaries

To define homology groups we must first focus on two particular types of chains, p-cycles and p-boundaries. A p-cycle c is a p-chain whose boundary is zero, $\partial_p c = 0$. The set of all p-cycles form a group denoted

$Z_p \subseteq C_p$. Z_p is a subgroup of C_p. Since Z_p is the set of all p-chains that go to zero under the pth boundary homomorphism, Z_p is the *kernel* of ∂_p denoted $Z_p = \text{ker}\partial_p$.

Definition 2.2 *Let G and H be groups and let $f : G \to H$ be a homomorphism from G to H. Let e_h denote the identity element in H. The kernel of f is defined as*

$$kerf = \{g \in G | f(g) = e_h\}$$

A p-boundary is a p-chain that is the boundary of a $(p+1)$-chain, $c = \partial_{p+1}d$ for $d \in C_{p+1}$. The set of all p-boundaries form a group denoted $B_p \subseteq C_p$. B_p is a subgroup of C_p. The group of p-boundaries is the *image* of the $(p+1)$-st boundary homomorphism, $B_p = \text{img}\partial_{p+1}$.

Definition 2.3 *Let G and H be groups and let $f : G \to H$ be a homomorphism from G to H. The image of f is defined as*

$$imgf = \{h \in H | \exists_{g \in G} f(g) = h\}$$

We are now ready to introduce the fundamental lemma of homology. Intuitively the following lemma states that if we take the boundary of a boundary we are left with nothing.

Lemma 2.4 $\partial_p \partial_{p+1} d = 0$ *for every interger p and every $(p+1)$-chain d.*

Proof:

$$\begin{aligned}
\partial_{p+1}(d) &= \sum_{i=0} (-1)^i d|[v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}] \\
\partial_p \partial_{p+1}(d) &= \sum_{j<i} (-1)^i (-1)^j d|[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_{p+1}] \\
&\quad + \sum_{j>i} (-1)^i (-1)^{j-1} d|[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1}] \\
&= 0
\end{aligned}$$

■

It follows from the fundamental lemma of homology that B_p is a subgroup of Z_p. The following figure illustrates the relationship between the groups we have defined. This figure is is courtesy of [1].

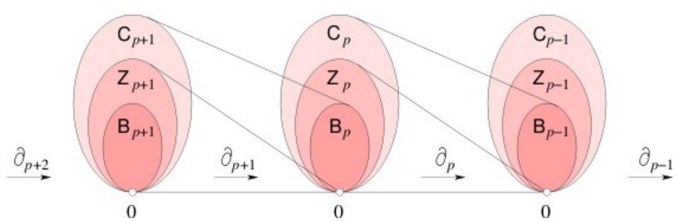

Figure 4: The relationships between C_p,Z_p and B_p via the boundary operator. The figure is courtesy of [1].

2.3 Homology Groups

Let γ_1 and γ_2 be p-chains. We say that γ_1 and γ_2 are homologous, denoted $\gamma_1 \sim \gamma_2$ if $\gamma_1 + \gamma_2 \in B_p$. Let $[\gamma] = \gamma + B_p$ be a coset of B_p formed by adding the chain γ to every element of B_p. Two chains are homologous if they are elements of the same coset. The *homology group* is the collection of all such cosets. The homology groups can be defined by taking quotients of the cycle and boundary groups. The p-th homology group is $H_p = Z_p/B_p$.

A cycle homologuous to zero is a cycle with trivial homology, and we say that this cycle *bounds*. For a *simple* 1-cycle on a surface M, it bounds if and only if it separates the surface M into two connected components. Recall that two loops are homotopic if one can deform into the other continuously. It turns out that homotopic implies homologuous, but not vice versa. That is, two homotopic cycles are also homologuous. Hence homology is a looser concept than homotopy.

A *generating set* A of a group G is a set of elements such that any element in G can be represented as a combination of elements in A and their inverse. If the group G is free, then there exists a *basis* A such that any element in G can be *uniquely* represented as a combination of elements in A and their inverse. A basis of a free abelian group G is also a smallest generating set for G, and the rank of G, denoted by rank(G) is the cardinality of any basis of G.

The groups C_p, Z_p and B_p are free abelian. Under modulo-2 addition, the quotient group H_p is also free abelian. Now let card(G) denote the cardinality, or *order*, of a group G. Since we are using modulo-2 addition, it is easy to see that card$(G) = 2^{\text{rank}(G)}$. Given a simplicial complex K, since the set of p-simplices form a basis for the group C_p, we have rank$(G) = n_p$, where n_p is the number of p-simplices in K. The number of cycles in a coset is the cardinality of B_p, so the cardinality of H_p is easy to compute.

$$\text{card}(H_p) = \text{card}(Z_p)/\text{card}(B_p)$$

Additionally the rank of H_p is just the difference.

$$\beta_p := \text{rank}(H_p) = \text{rank}(Z_p) - \text{rank}(B_p)$$

Here β_p is the pth *Betti number* of the simplicial complex K. Informally the pth Betti number is the number of unconnected p-dimensional surfaces. The first few Betti numbers have an easy intuitive meaning.

1. β_0 is the number of connected components.
2. β_1 is the number of two-dimensional or "circular" holes.
3. β_2 is the number of three-dimensional holes or "voids".

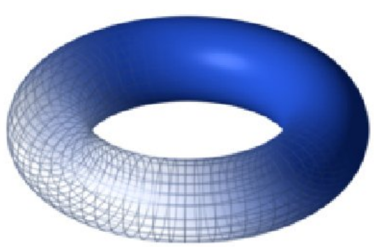

Figure 5: The torus has one connected component, two circular holes (the one in the center and the one in the middle of the tube), and one void (the inside of the tube). These give Betti numbers 1,2,1.

References

[1] H. Edelsbrunner and J. Harer. *Computational Topology: An Introduction*. 2009.

[2] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.