Lecture 5: Simplicial Complex

2-Manifolds, Simplex and Simplicial Complex

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First part of this lecture finishes 2-Manifolds. Rest part of this lecture talks about simplicial complex.

1 2-Manifolds

1.1 Comparing Curves

Given two curves P, Q on a 2-manifold M. Measure the similarity between P and Q is an interesting problem. Traditional metric, like Hausdroff distance, is not suitable here. As can be seen in Figure 1, the Hausdroff distance between P and Q can be very small while they are of great difference. We need other similarity measurements.

Fréchet distance is a good similarity measurement for curves in Euclidean space. It can be simply described by a daily example. Suppose a dog and its owner are walking along two different paths (curves), connected by a leash. Both of them are moving continuously and forwards only, at any speed or even stop. Then length of the shortest leash is the Fréchet distance between the two paths (curves). Fréchet distance is a good similarity measurement for curves in Euclidean space. For curves on surfaces, distance between two points can be hard to compute.

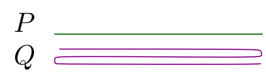


Figure 1: Two greatly different curves have a small Hausdroff distance

Another way is using the minimal area swept when continuously deforming P and Q into one as their distance. However, it has the same problem as Fréchet distance when handling curves on surfaces. This measurement also requires P and Q to share the same start point and end point, i.e., they form a loop. If they do not, we can connect their start points and end points, respectively, to get a loop.

To handle curves on surfaces, we can map them to a space where similarity measurements like Fréchet distance or minimal deformation area are easy to be computed. Recall that the universal cover of M is planar, which is computation friendly to Fréchet distance and minimal deformation area. We can map P and Q to the universal cover of M, and use the distance between their images to measure the similarity of P and Q.

However, not any two curves on M can be measured in this way. For example, as Figure 2 shows, P and Q can not be deformed to each other, because they are not homotopic.

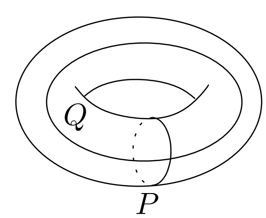


Figure 2: P and Q can not be continuously deformed to each other.

1.2 Homotopy

Definition 1.1 *Let* X *and* Y *be two topological spaces. Two continuous maps* $f, g : \mathsf{X} \to \mathsf{Y}$ *are* homotopic *if there exist a continuous map* $H : [0, 1] \times \mathsf{X} \to \mathsf{Y}$ *such that* $H(0, \cdot) = f$ *and* $H(1, \cdot) = g$*, denoted as* $f \sim g$*. The continuous map* H *is called a* homotopy *between* f *and* g*.*

Consider $[0, 1]$ as a time interval, f and g are homotopic means we can find a family of maps H such that f and g are continuously deformed: at time 0 we have f, and at time 1 we have g.

Definition 1.2 *Two topological spaces* X *and* Y *are* homotopic equivalent *if there exist two continuous maps* $f : \mathsf{X} \to \mathsf{Y}$ *and* $g : \mathsf{Y} \to \mathsf{X}$*, such that* $f \circ g \sim id_{\mathsf{Y}}$ *and* $g \circ f \sim id_{\mathsf{X}}$*.*

Definition 1.3 *Given a topological space* X*, and its subspace* $A \subseteq \mathsf{X}$*. A continuous map* $f : [0,1] \times \mathsf{X} \to A$ *is a* deformation retraction *from* X *onto* A *if for any* $x \in \mathsf{X}$ *and any* $a \in A$*, there is* $f(0,x) = x$*,* $f(1,x) \in A$ *and* $f(1,a) = a$*.* A *is called a* deformation retract *of* X.

We can consider a deformation retraction f as a continuous deformation process squeezing X to A. A well known example of deformation retraction is to deforming an annulus to a cycle, as Figure 3 shows. All points in the annulus moves straightly towards center until reach the inner cycle boundary. Notice that this deformation retraction is irreversible.

2 Simplicial Complex

Simplicial complex provides an good way for representing topological structures in computers. It is also an interesting topic of algebraic topology due to its combinatorial nature. In following lectures, we will know simplicial complex is the basis of simplicial homology. A simplicial complex is built by gluing its blocks, called simplex, together.

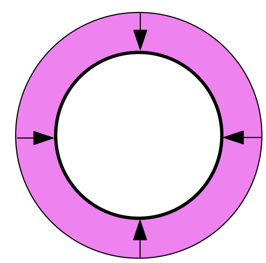


Figure 3: Deforming an annulus to a cycle

2.1 Simplex

A given set of points $\{p_0, p_1, ..., p_d\} \subseteq \mathbb{R}^d$ is *geometrically independent* if the set of vectors $\{p_i - p_0\}$ is *linearly independent*. Here, two vectors v_i, v_j are linearly independent if there does not exist two constant a_i, a_j such that $a_i v_i + a_j v_j = 0$ and either a_i or a_j should be non-zero. In $\mathbb{R}^d$ we can have at most $d-1$ geometrically independent points. A combination $x = \sum_{i=0}^{d} a_i p_i$ is a *convex combination*, if $\sum_{i=0}^{d} a_i = 1$ and all a_i are non-negative. The *convex hull* of a given point set $\{p_0, p_1, ..., p_d\}$ is the set of all convex combinations, denoted as CH$(\{p_0, p_1, ..., p_d\}) = \{\sum_{i=0}^{d} a_i p_i | \sum_{i=0}^{d} a_i = 1 \text{ and } a_i \geq 0\}$.

Definition 2.1 *A* d-simplex σ *is the convex hull of* $d+1$ *geometrically independent points* $\{p_0, p_1, ..., p_d\}$*, i.e.,* $\sigma = CH(\{p_0, p_1, ..., p_d\})$*. We also say the point set* $\{p_0, p_1, ..., p_d\}$ spans σ. d *is called* dimension *of* σ*, denoted as* $\dim \sigma = d$*.*

First few low dimensional simplices have their own names: 0-simplex, 1-simplex, 2-simplex and 3-simplex are also called vertex, edge, triangle and tetrahedron, respectively. Figure 4 shows them.

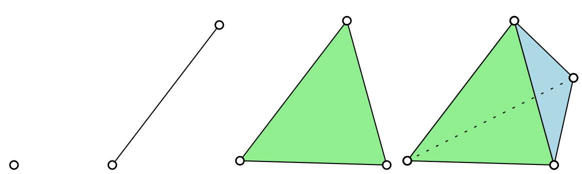


Figure 4: From left to right show a vertex, an edge, a triangle and a tetrahedron

Any non-empty subset A of a point set $\{p_0, p_1, ..., p_d\}$ spans a simplex $\sigma_A \subseteq \sigma$. σ_A is called a *face* of σ. If A is a proper subset, then σ_A is called a *proper face* of σ. The *boundary* of σ, denoted as $\partial\sigma$, is the union of all proper faces of σ.

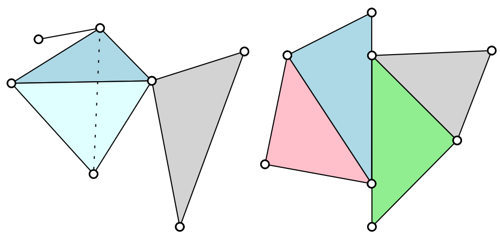
From its definition, we know a d-simplex σ must be convex. It also has many good topological properties: σ is homeomorphic to a $\mathbb{B}^d$; its boundary is homeomorphic to $\mathbb{S}^{d-1}$; and the interior of σ is homeomorphic to $\mathbb{R}^d$, i.e., $\sigma \setminus \partial\sigma \approx \mathbb{R}^d$. Actually, all d-simplex are homeomorphic.

Definition 2.2 *Let σ be a d-simplex defined on $\{p_0, p_1, ..., p_d\}$, then any point $p \in \sigma$ can be represented as $p = \sum_{i=0}^{d} a_i p_i$, where $a_i \geq 0$ and $\sum_{i=0}^{d} a_i = 1$. The vector $\langle a_0, a_1, ..., a_d \rangle$ is called the* barycentric coordinate *of p.*

2.2 Simplicial Complex

Definition 2.3 *A* simplicial complex *K is a collection of simplices such that*
(1) *If $\sigma \in K$, then for any face σ' of σ, we have $\sigma' \in K$;*
(2) *For two simplices $\sigma_1, \sigma_2 \in K$, $\sigma_1 \cap \sigma_2$ is either $\emptyset$ or a face of both σ_1 and σ_2.*

From its definition, we can see K is a combinatorial representation of a topological space. All simplices in K meet nicely and have no improper intersection in K. The *dimension* of K is the highest dimension of any of K's simplices. The set of simplices shown in Figure 5a is a simplicial complex, whereas the one in Figure 5b is not a simplicial complex.



(a) (b)

Figure 5: (a) A simplicial complex. (b) Not a simplicial complex

Definition 2.4 j-skeleton *of K, denoted as $K^{(j)}$, is the collection $\{\sigma \in K | dim\ (\sigma) \leq j\}$.*

From the definition we know 0-skeleton is the set of vertices, 1-skeleton is the set of vertices and edges, 2-skeleton is the set of vertices, edges and triangles and so on.

Definition 2.5 *A* subcomplex *of K is a subset $K' \subseteq K$ which itself is still a simplicial complex.*

Topological spaces studied in previous lectures are continuous. However, from the above discussion we can see simplical complex is discrete. The *underlying space* $|K|$ of K, which is defined as $|K| = \{x \in \sigma | \forall \sigma \in K\}$, associates the discrete simplical complex with continuous topological space. Two simplicial complices are equivalent if their underlying space are homeomorphic.

The *star* of a 0-simplex $v \in K$ is defined as the set of all simplices having v as a face, denoted as $St(v) = \{\sigma \in K | v \in \sigma\}$. Notice that v is a face of itself, so $v \in St(v)$. The *closed star* (or *closure*) of v, $\overline{\text{St}}(v)$, is the smallest subcomplex of K containing $St(v)$. The *link* of v, $\text{Lk}(v)$, is defined as $\text{Lk}(v) = \overline{\text{St}}(v) - \text{St}(v)$. $St(v)$ can be considered as a open neighborhood of v in topological space, then $\overline{\text{St}}(v)$ corresponds to the concept of closed neighborhood and $\text{Lk}(v)$ is the boundary of $St(v)$. Figure 6 show the star, closed star and link of a vertex. Notice that the concepts of star, closed star and link for a 0-simplex are just special cases of a set of d-simplices. Please refer to [1] for their definitions.

Definition 2.6 *An* abstract simplicial complex *is a finite collection of sets K such that if a set $\sigma \in K$, then for any subset $\sigma' \subseteq \sigma$, $\sigma' \in K$.*

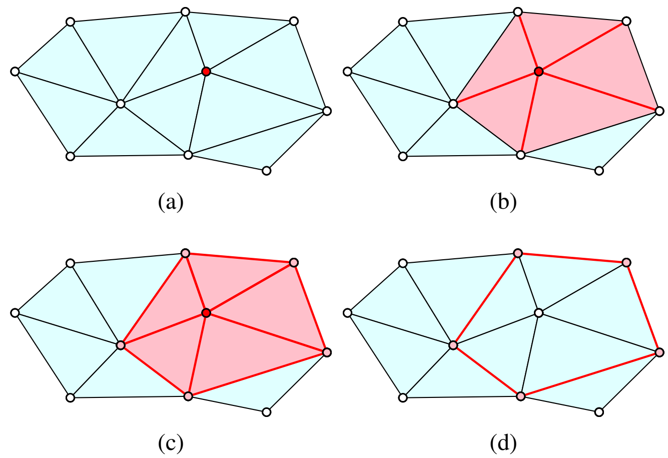


Figure 6: (a) A vertex v. (b) $St(v)$. (c) $\overline{\text{St}}(v)$. (d) $\text{Lk}(v)$.

Loosely speaking, abstract simplicial complex is a simplicial complex without the associated geometric information. It it pure combinatorial. With embedded geometry, an abstract simplicial complex K' becomes a *geometric simplicial complex* K, and such K is called a *geometric realization* of K'. K can be embedded in Euclidean space. The following theorem tells us abstract simplicial complex does not does not mess up geometric information.

Theorem 2.7 *Any abstrct d-dimensional simplicial complex has a geometric realization in Euclidean space* $\mathbb{R}^{2d+1}$.

Proof:

Let K be a d-dimensional abstract simplicial complex, f be an injection $f : K \to \mathbb{R}^{2d+1}$. We need to prove $f(K)$ is a geometric realization of K. Obviously, f preserves the topological structure of K, i.e., if a simplex $\sigma \in K$ and $\tau \in \sigma$, then $f(\tau) \in f(\sigma) \in f(K)$. We only need to show there is no improper intersection in $f(K)$.

Let σ_0 and σ_1 be two simplices in K with $a_0 = \text{dim } (\sigma_0), a_1 = \text{dim } (\sigma_1)$. We have cardinality of union of the two simplices is card $(\sigma_0 \cup \sigma_1) =$ card $(\sigma_0) +$ card $(\sigma_1) \leq a_0 + 1 + a_1 + 1 \leq 2d + 2$. This means points in $f(\sigma_0 \cup \sigma_1)$ are geometrically independent. Then any convex combination of points in $f(\sigma_0 \cup \sigma_1)$ has a unique barycentric coordinate.

Consider f is injective, we have $f(\sigma_0) \cap f(\sigma_1) = f(\sigma_0 \cap \sigma_1)$. If $f(\sigma_0) \cap f(\sigma_1) \neq \emptyset$, then any convex combination $x \in f(\sigma_0) \cap f(\sigma_1)$ iff x is a convex combination of $f(\sigma_0 \cap \sigma_1)$. That is to say, $\text{CH}(f(\sigma_0) \cap f(\sigma_1)) = \text{CH}(f(\sigma_0 \cap \sigma_1))$. We already know $\text{CH}(f(\sigma_0 \cap \sigma_1))$ is a simplex, so the intersection of $f(\sigma_0)$ and $f(\sigma_1)$ is either empty or a simplex. This proves there is no improper intersection in $f(K)$.

In sum, $f(K)$ is a geometric realization of K. ■

By this theorem, we know a Klein bottle can be embedded in $\mathbb{R}^5$. Actually, it can be embedded in $\mathbb{R}^4$, so the bound is not tight.

2.3 Push operation

We already know deformation retraction does not change topological structure (homotopy type) while transforming one topological space to another. There is a similar process for simplicial complex.

Given a simplicial complex K, a simplex $\sigma \in K$ is *free* if $\sigma \in \partial K$. Push free faces of σ towards its non-free faces, under certain conditions, is essentially a deformation retraction.

Definition 2.8 Push operation *for a d-dimensional simplicial complex continuously deforms the set of free faces of a simplex onto the set of its non-free faces, if both sets are homeomorphic to* $\mathbb{B}^{d-1}$.

Push operation forms a deformation retraction of the input simplex. Performing a sequence of push operations to the input simplicial complex does not change its homotopy type, and can be used to somewhat "simplify" a simplicial complex.

References

[1] H. Edelsbrunner and J. Harer. *Computational topology: an introduction*. 2009.