Lecture 4: 2-Manifolds (Continued)

Universal Cover, Paths and Loops, the First Fundamental Group

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1 Universal Cover

Universal Covers help us in studying complex spaces by lifting them onto simpler spaces in such a way that local properties of the space are preserved.

Definition 1.1 *A* **Covering** *of a space* X *is another space* C *together with a map* $\Phi : C \to \mathsf{X}$ *such that for any point* $x \in \mathsf{X}$*, there exists an open neighborhood* U *of* x *such that* $\Phi^{-1}(U)$ *is a disjoint union of open sets in* C*, each of which is mapped homeomorphically onto* U*.*

Lets look at some examples to get an intuitive idea of what a Covering is.

Example 1: Let $\mathsf{X} = \mathbb{S}^1$ and C be an infinite family of disjoint circles as shown in Figure 1 on the left. Here, Φ simply maps a point on a circle in C onto the corresponding point in X. Hence, for a point $x \in \mathsf{X}$, its open neighborhood Us pre-image in C is a disjoint union of open sets such that each such pre-image is mapped homeomorphically onto U.

Example 2: Let $\mathsf{X} = \mathbb{S}^1$ and C be a helix in $\mathbb{R}^3$, along with Φ as shown in Figure 1 on the right. Again, a point on X and its pre-image are locally same.

Example 3: Thinking of the helix in Example 2 as a coiled up real line $\mathbb{R}$, let $\mathsf{X} = \mathbb{S}^1$, $C = \mathbb{R}$ and $\Phi(t) = (cos(t), sin(t))$. This gives yet another covering of $\mathbb{S}^1$, this time with $\mathbb{R}$.

Trivially, each space is a covering of itself through the identity map.

Observation 1.2 *A covering is locally homeomorphic, but there can be multiple global mappings for the same point.*

Definition 1.3 *A* **Universal Cover** *of* X *is a covering space that is* **simply-connected**.

For any 2-manifold, its universal cover is either $\mathbb{R}^2$ or $\mathbb{S}^2$.

Definition 1.4 *A space is* **simply-connected** *if any loop in the space can be continuously deformed into a single point.*

For example, $\mathbb{R}^2$ and $\mathbb{S}^2$ are simply-connected.

Example 1: $\mathbb{S}^2$. Its universal cover is $\mathbb{S}^2$ itself through the identity map.

Example 2: Torus. Recall that A torus can be cut along two special loops and unrolled into a square, as shown in Figure 2. Hence, one can map the rectangle onto the torus. This map is locally homeomorphic everywhere except on the boundary of the rectangle. The problem can be solved by gluing copies of the rectangle as shown in Figure 3. But now, the mapping is not locally homeomorphic at the boundary of this $+$ shape, but we can keep gluing copies of the rectangle along the boundary indefinitely to get a tiling of $\mathbb{R}^2$. Hence, $\mathbb{R}^2$, along with the tiling based mapping Φ onto the torus gives the universal cover of the torus.

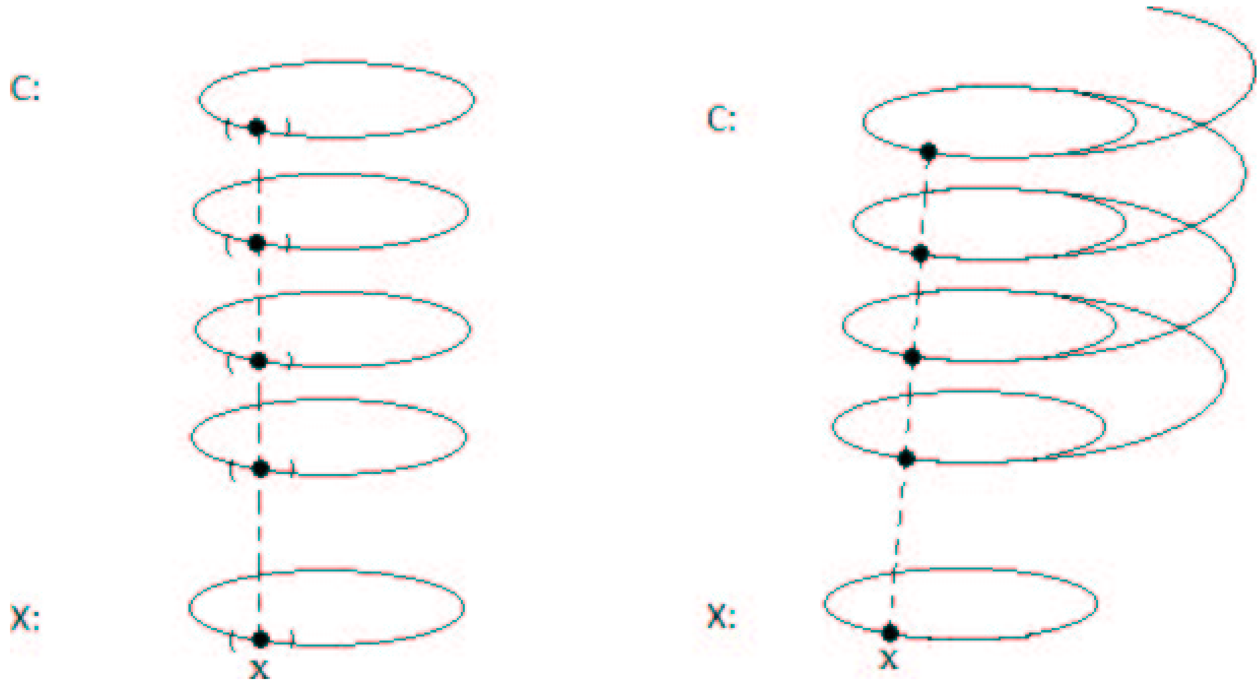

Figure 1: Coverings for different spaces

Any point on the torus will have infinitely many pre-images in $\mathbb{R}^2$ (one in each rectangle in the tiling). Lets look at pre-images of some cycles on the torus. Figure 4 on left shows the pre-image of the black loop on the torus. Note that the pre-image of the closed loop is open. Also, the pre-image of the red and blue loops are infinitely many parallel red and blue lines in $\mathbb{R}^2$. This is not true for all the loops. Figure 4 on right shows the pre-image of another loop on the torus. Although there are infinitely many copies in the pre-image, each such pre-image is a closed loop.

Observation 1.5 *This process can be generalized to any* 2*-manifold by using its polygonal schema as a unit for tiling* $\mathbb{R}^2$.

Theorem 1.6 *Any* 2*-manifold adopts a universal covering space.*

Proof: As seen in Example 1, the universal cover of $\mathbb{S}^2$ is $\mathbb{S}^2$ itself. For other 2-manifolds, the universal covering is given by $\mathbb{R}^2$. ■

Finally, given a universal cover $\phi : C \to X$ of X, ϕ is many to one, so the inverse of a point $x \in X$ is a set of disjoint point in C. If we fix one of its pre-image, call it the lift of this point x. Once the lift of x is fixed, a path with x being the starting endpoint then has a unique lift.

2 Paths, Loops and the Fundamental Group

Recall that a path is a map $r : [0.1] \to \mathsf{M}$. It is closed if $r(0) = r(1)$. A path also has a natural orientation going from $r(0)$ to $r(1)$. Reversing this orientation gives the inverse of the path, as shown in Figure 5.

Consider the six loops on a 3-torus as shown in Figure 6. r_1 and r_4 look similar, as do r_2 and r_3. This is because r_1 can be deformed to r_4 and r_2 to r_3 on the surface of the triple-torus. r_1 and r_2, however, are different as we cannot continuously deform one to other along the surface without tearing the loop open.

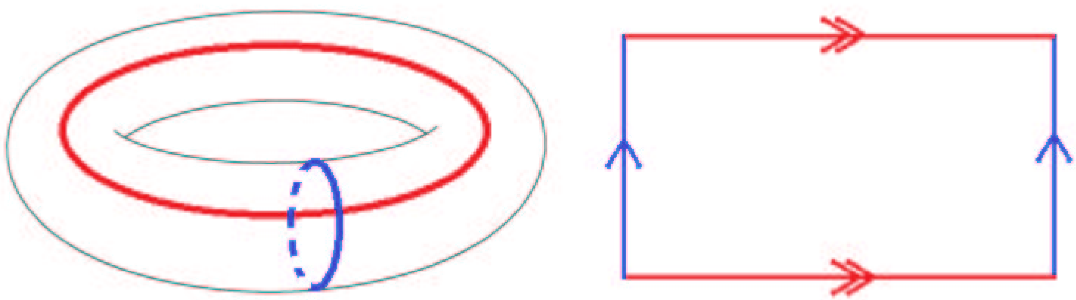

Figure 2: Cutting a torus to get a rectangle

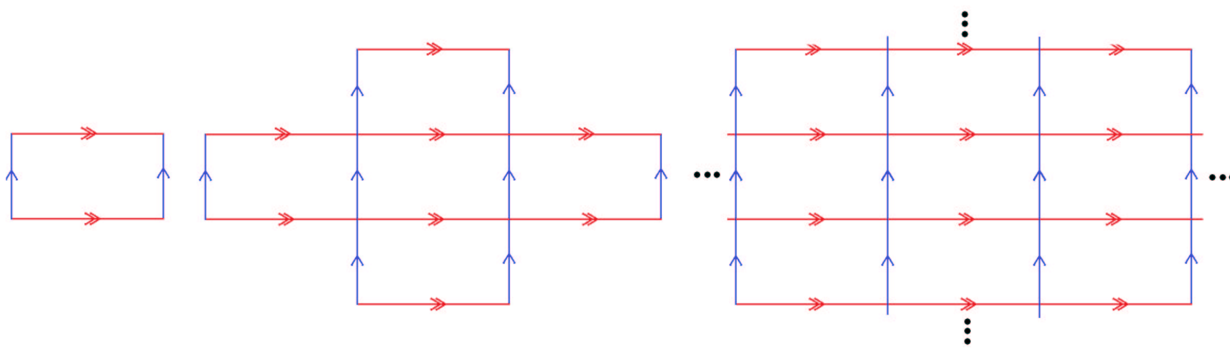

Figure 3: Gluing copies to get a universal cover

Similarly, r_5 is not similar to either r_1 or r_2. r_5 and r_6 may look similar, but again, we cannot continuously deform one to the other. Hence, There are different kinds of loops and paths on a surface, and this section aims at studying and classifying them.

Definition 2.1 *Two paths r_1 and r_2 with same end-points are homotopy equivalent (or homotopic) if there exists a map (or a family of curves) $H : [0,1] \times [0,1] \to \mathsf{M}$ such that*

$H(0, \cdot) = r_1(\cdot)$
$H(1, \cdot) = r_2(\cdot)$
and H is continuous. Homotopy equivalence is denoted by $r_1 \simeq r_2$.

The first variable in $H(\cdot, \cdot)$ parametrizes the family of curves, while the second variable is the parameter for the curve itself. (See Figure 7)

In $\mathbb{R}^2$, and on $\mathbb{S}^2$, any two curves with same end points are homotopic. The homotopy relation is transitive, i.e. $a \simeq b, b \simeq c \implies a\ c$. It is also reflective ($a \simeq b \implies b \simeq a$). Furthermore, $a \simeq a$. Hence this relation is an equivalence relation. Therefore, curves on a surface can be clustered into equivalence classes based homotopy. This gives a family $[r]$ of equivalence paths for a path r, called its **homotopy class**.

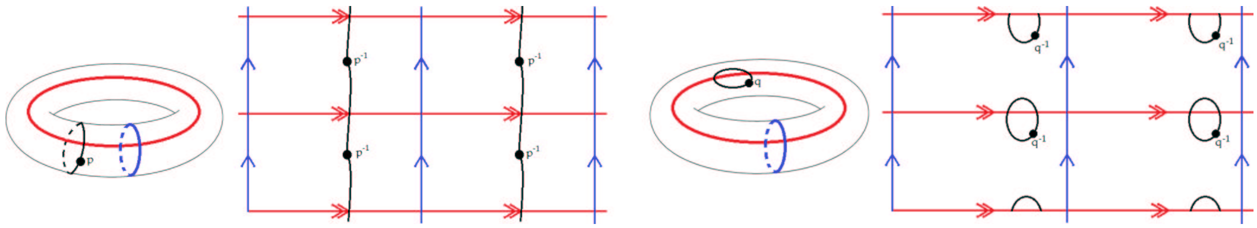

Figure 4: Pre-images of different types of loops on a torus

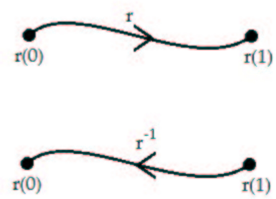

Figure 5: Orientation of a path

Definition 2.2 *A loop is* **contractible** *(or null-homotopic) if it is homotopic to a point (or it has a point in its homotopy class).*

Consider two loops passing through a single point. We can define concatenation of two loops $r_1 \circ r_2$ using the orientation of the two loops. The concatenated loop is traversed first along r_1 and then along r_2. This can be extended to equivalence classes.

Definition 2.3 *For two equivalence classes α and β, define their concatenation as $\alpha \circ \beta = [r_1] \circ [r_2] = [r_1 \circ r_2]$ where r_1 and r_2 are two randomly chosen loops from α and β, respectively.*

Note that the choice of loops does not affect the equivalence class of concatenation, since any other loops chosen from α and β can be continuously deformed to r_1 and r_2.

Now, recall that a group is a set S together with an operator $\cdot$, such that

- $\forall a, b \in S$, $a \cdot b \in S$
- $\forall a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $\exists i \in S$ such that $a \cdot i = i \cdot a = a$
- $\forall a \in S$, $\exists a^{-1} \in S$ such that $a \cdot a^{-1} = a^{-1} \cdot a = i$

The set of homotopy equivalence classes along with the concatenation operator defined earlier forms a group. The identity element if the contractible class. The inverse of an equivalence class can be obtained by choosing a random path from the class, reversing its orientation and finding the equivalence class of this reversed path. Intuitively, one goes from $r(0)$ to $r(1)$ and comes back along the same path, making it a loop which is also contractible.

This group is called the first fundamental group, denoted $\pi_1(\mathsf{M})$ with respect to a fixed base point (say, x). However, for path-connected spaces, the choice of base-point does not matter. If we decide to use a new base-point (say, y), we can get a 1-to-1 correspondence between loops using base-point x and those

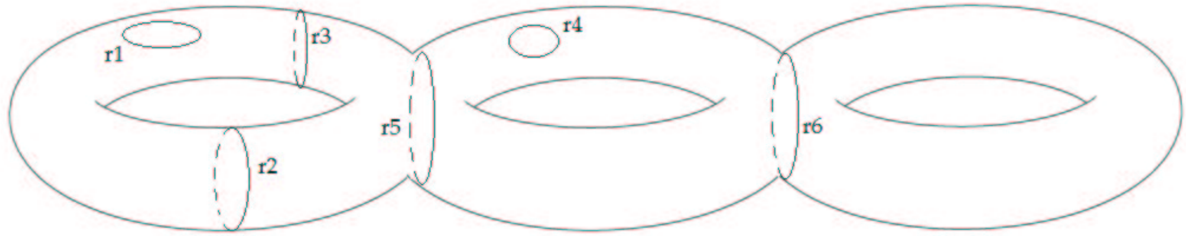

Figure 6: Loops on a 3-torus

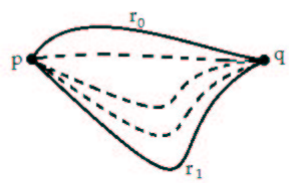

Figure 7: Homotopy equivalence of two paths

using y, since there is always a path from x to y. Hence, for path-connected spaces, we can discuss the first fundamental group without explicitly specifying a base point.

A space with a trivial fundamental group is called *simply connected*. In other words, any closed curve is contractable. The famous Poincaré conjecture is basically that:

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

This statement is true for 2-manifolds as the theorem above suggested. It holds for dimension higher than 4 (in which case we ask whether a homotopy sphere is homeomorphic to a sphere, not just the first fundamental group). The answer for this claim is only very recently settled by Russian mathematician Perelman.

3 Generators of $\pi_1(\mathsf{M})$

For a 2-manifold M of genus g, the first fundamental group has a free generator consisting of $2g$ loops. Any homotopy class of M can be written as a linear combination of the homotopy classes of these loops and their inverse. Furthermore, it turns out that cutting the 2-manifold along these $2g$ loops will give us a topological disk; in fact, it will give us a polygonal schema of the manifold.

Claim 3.1 *Any loop whose image is contained completely within the polygonal schema of the manifold is contractible.*

Any loop which lies completely inside the polygonal schema can be continuously deformed onto the boundary of the schema which, after gluing, becomes a single point on the surface. Image of a generator will be a single edge of the boundary of the schema, make them non-contractible. In fact, any loop on the surface can be obtained by suitably combining the generator loops (hence the name).

4 Test for homotopy type

1. Simple closed curve

For a simple closed curve (i.e. with no self intersections), we can use the triangulation of the surface to quickly determine if a loop is contractible. Note that for a simple curve to be contractible, it has to enclose a disk (the curve can then contract continuously within the disk to a point). The test is as follows:

From the surface triangulation, remove all the triangles that intersect the curve. This will create one or more disjoint connected components. Use the triangulation of the components to quickly determine its Euler characteristic (= ($\#$ vertices) - ($\#$ edges) + ($\#$ triangles)). If any component has an Euler characteristic of 1 (i.e. that of a disk), then the curve can deform to a point within that disk and hence the curve is contractible, otherwise not.

2. Self-intersecting curves

For closed curves that are not simple, we have to go to the universal cover. We had previously observed that pre-images of some loops were also closed loops while for others they were paths. The contractible curves are precisely the ones whose pre-images were also closed, while non-contractible loops are the ones with open paths as pre-images. Since there are infinitely many pre-images, we arbitrarily chose one pre-image to work with. Next, we choose a point on the curve and start tracing its pre-image in the universal cover. We stop when we traverse the entire loop once and return to the chosen starting point on the curve. This is called a *lift* of the curve. If the pre-image of the ending point also ends up at the pre-image of the starting point, the curve is contractible. Otherwise, it is not.

Lemma 4.1 *A loop in M is contractible if and only if its lift is closed in the universal cover.*

Refer to Figure 4. The curve on left is not contractible, while the one on right is.