Lecture 3: 2-Manifolds

*Surface Classification, Polygonal Scheme, Euler-Characteristic*

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1 Surface Classification

2-manifolds, often referred to as *surfaces*, are of special interest, as they appear most often in real life, especially in graphics. In this class and next class, we will focus on 2-manifolds, talking about how to classify them, some topological quantities for surfaces and their computation.

1.1 Orientable/Non-Orientable Surface

![Figure 1: (a) Cylinder; (b) Möbius strip](image)

Suppose there is a curve traversing parallel to the boundary of a Möbius strip, and there’s a point \( p \) moving along this curve in one direction, at \( s \) position, we assign \( p \) a coordinate frame with the normal pointing outside the paper. We find that at first time \( p \) go back to \( s \), the frame normal turns out to point inside the paper. However, if this point continues its traverse, after another circle, the normal go back to the initial direction. We call the curve between \( s \) and \( p \)'s first arrive to \( s \) an orientation-reversing closed curve; and the curve between \( s \) and \( p \)'s second arrive to \( s \) as orientation-preserving closed curve.

**Definition 1.1** A surface is orientable if any closed curve insides is orientable preserving; Otherwise, it’s non-orientable.

For example, Möbius strip is a non-orientable surface with one boundary. A cylinder (Figure 1(a)) is an orientable surface with two boundaries. A nature question now rise is what should an non-orientable surface without boundary look like? The simplest non-orientable surface without boundary is a projective plane, denoted by \( \mathbb{P}^2 \).

Suppose there is a circle (Figure 2 left) with angle changing from 0 to \( 2\pi \), and there is a point \( p \) on the circle with angle \( \alpha \), the corresponding antipodal \( p' \) is a point with angle \( \pi - \alpha \). Projective plane \( \mathbb{P}^2 \) is obtained by identifying each pair of antipodal points \( (p, p') \) on the sphere (Figure 2 right). A \( \mathbb{P}^2 \) also can be obtained by gluing a Möbius strip to a disk, due to the fact that a Möbius strip has a boundary which is homeomorphic to a circle. The operation of gluing a Möbius strip to another surface is often referred to adding a *cross-cap*. Similarly, glue two Möbius strip together, we get a Klein bottle. Both \( \mathbb{P}^2 \) and Klein bottle can not be embedded in \( \mathbb{R}^3 \), that’s also the reason why they are always drawn with self-intersections.
1.2 Connected Sum

*Connected sum*, denoted by #, is an operation to construct a new surface based on current surfaces. The procedure can be described as following: Given two surfaces $M$ and $N$, we move an open disk $D$ on both of them. Then we glue $M - D$ and $N - D$ together along the boundary of $D$, and denote the new surface as $M \# N$. For example, let $T^2$ be a torus, $T^2 \# T^2$ is a double torus; $S^2 \# \mathbb{P}^2 = \mathbb{P}^2$, because a sphere with a removed open disk is a closed disk and a $\mathbb{P}^2$ with a removed open disk is a Möbius strip. By the same reason, $\mathbb{P}^2 \# \mathbb{P}^2$ is a Klein bottle. Note the fact that a new surface obtained by connecting an orientable surface with a non-orientable one will still be a non-orientable one, since it’s always possible for us to find an orientation reversed closed curve(s) on the result surface.

1.3 Classification Theorem

**Definition 1.2** Cover: A collection $A$ of subsets of a space $X$ is said to cover $X$, or to be a covering of $X$, if the union of the elements of $A$ is equal to $X$. It is called an open covering of $X$ if its elements are open subsets of $X$.

**Definition 1.3** Compact: A space $X$ is said to be compact if every open covering $A$ of $X$ contains a finite subcollection that also covers $X$.

Take interval as an example. An open interval is not compact; but the closed one is. The only non-compact surface is a plane.

**Definition 1.4** Totally bounded: A metric space $(X, d)$ is said to be totally bounded if for every $\epsilon > 0$, there is a finite covering of $X$ by $\epsilon$–balls (a ball with $\epsilon$ as the radius).

Note that totally boundedness implies boundedness.

**Claim 1.5** Let $(X, d)$ be a metric space, then it is compact if and only if it is closed and totally bounded.

**Classification of surface without boundary**: On the perspective of orientable and non-orientable, the two infinitive families of compact, connected surfaces are

1. The family of orientable surface: $S^2$, $T^2$, $T^2 \# T^2 \# \ldots \# T^2$

2. The family of non-orientable surface: $\mathbb{P}^2$, $\mathbb{P}^2 \# \mathbb{P}^2 \# \ldots \# \mathbb{P}^2$

Note $T^2 \# \mathbb{P}^2 = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$. Therefore, above two families exhaust all possible 2–manifolds without
boundary. In the family of orientable surface, \( \mathbb{T}^2 \# \mathbb{T}^2 \# \ldots \# \mathbb{T}^2 \) is a \( g \)-genus surface (it has \( g \) handles). Similarly, \( \mathbb{P}^2 \# \mathbb{P}^2 \# \ldots \# \mathbb{P}^2 \) is a surface with \( k \) cross-caps.

2 Polygonal Schema

We are now going to talk about another way to look at all closed 2-manifolds, which can greatly help us to understand topology of 2-manifolds w/o boundaries. Recall that we have used a rectangle and its stitching pattern to represent a Mobius band, or a torus before. This rectangle is very useful, and provide us a flattening of the surface onto the plane. We call such a simple convex polygon together with the gluing pattern for the boundary a **polygonal schema** of a surface. Note the polygonal schema is not unique. Figure 3 shows two polygonal schemas for \( \mathbb{P}^2 \).

Each polygonal schema has a corresponding naming scheme (labeling scheme), the left one in Figure 3 is \( aa \); and the right one is \( abab \). Let \( \mathbb{T}^2 \) be a torus with naming scheme \( aba^{-1}b^{-1} \), Figure 4 shows the main procedure of gluing two \( \mathbb{T}^2 \)s to be a double torus name \( aba^{-1}b^{-1}aba^{-1}b^{-1} \). Such procedure can be continuously repeated to generate \( g \)-genus orientable surface with name

\[
aba^{-1}b^{-1}a_2b_2a_2^{-1}b_2^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1}
\]

Similarly, \( \mathbb{P}^2 \# \mathbb{P}^2 \# \ldots \# \mathbb{P}^2 \) projective plane \( a_1a_1a_2a_2 \ldots a_ka_k \) can be obtained by connecting \( k \) \( \mathbb{P}^2 \) together. We call these polygonal schema for all compact surfaces **canonical polygonal schema**. Note that...
Klein bottle has a more intuitive polygonal scheme whose naming scheme is $aba^{-1}b$.

There are different reasons why polygonal schema is very useful. For example, we can obtain a triangulation of the underlying manifold by triangulating the polygonal schema. We will see one of its uses later when we talk about triangulations. Intuitively, this helps to map an arbitrary surface to some planar domain, which we will see later when we talk about the so-called universal cover. Note that other than those points on the stitching lines, every point is mapped to the original surface homeomorphically.

3 Triangulation

Given a surface $M$, a triangulation $K$ of $M$ is a decomposition of $M$ into cells such that
(i) each cell is homeomorphic to a disk
(ii) boundary of each cell has 3 arcs
(iii) any two cells intersect at a node or an arc
If a surface can adapt a triangulation, we call it triangulable. Clearly, if some surface is triangulable, it has to be homeomorphic to its triangulation. The condition of homeomorphism requires that any two triangles are either disjoint, share an edge, or share a vertex (condition(iii)).

3.1 Orientability checking

Given a triangulation of a surface, we may orient each triangle. Two triangles sharing an edge are consistently oriented if they induce opposite orientations on the shared edge. This surface is called orientable if and only if the triangles can be arranged in following way: every adjacent pair is consistently oriented [1].

The algorithm used for orientability checking can be finished in $O(N)$, where $N$ is the totally number of triangles, edges and vertices on the triangulated surface model.

3.2 Euler Characteristic

Suppose we have a surface with a triangulation $K(V, E, F)$, the Euler characteristic of $K$ is $\chi = |V| - |E| + |F|$. Note $\chi$ is intrinsic to the domain(surface) rather than the triangulation.

Claim 3.1 Any triangulation of a plane gives rise to $\chi = |V| - |E| + |F| = 2$.

Similarly, $\chi(S^2) = 2; \chi(T^2) = 0; \chi(P^2) = 1$. In fact, the formula $|V| - |E| + |F|$ works for any cell decomposition, not necessarily triangulations. The only constraint is that each cell is a topological disk, and an arc is a path between two vertices such that there is no other vertex in the interior. What does this imply? First, any two homeomorphic manifolds have the same Euler Characteristic, because the canonical polygon is one simplest cell decomposition. Second, we actually can compute it for any type of surface. Example: for a torus, from its canonical polygonal schema, it is easy to see that there are 1 vertex, 2 edges and 1 face. Hence its Euler Characteristic is $1 - 2 + 1 = 0$. Using the polygonal schema, we can compute this for any compact surface:

Lemma 3.2 An orientable 2-manifold with $g$ genus has Euler Characteristic $2 - 2g$, and a non-orientable 2-manifold with $g$ genus has Euler Characteristic $2 - g$. Given a 2-manifold, its Euler Characteristics and orientability uniquely decides its topology. Furthermore, for each additional hold, the Euler Characteristic decreases by one, given $\chi = 2 - 2g - h$ in the orientable case and $\chi = 2 - k - h$ in the non-orientable case [1].

It is also easy to verify the following claim from the definition of connected sum:

Claim 3.3 Given the connected sum $M \# N$ of two surfaces $M$ and $N$, $\chi(M \# N) = \chi(M) + \chi(N) - 2$. 


Identify topology type of a surface. This also gives us an algorithm to compute the topology type of a 2-manifold from its triangulation. Basically, first, we test its orientability by walking through the triangulation. If all are consistently oriented, then the surface is orientable. Otherwise, the surface is non-orientable. Next, we compute its Euler Characteristics and obtain \( g \) depending on whether it is orientable or non-orientable.

References
