Lecture 14: Morse-Smale Complex

Topics in Computational Topology: An Algorithmic View

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1 Introduction

Consider a drop of water placed at an arbitrarily chosen point on some 2-manifold surface. The drop will inevitably begin to move along the gradient of the surface, towards a local minimum. For the purposes of this discourse, assume that the momentum of the drop and any electrostatic interactions between the drop and its surroundings are negligible. Then the drop will at each point on its path proceed along the gradient evaluated at that point until it reaches a local minimum, where it will come to rest, and the path taken by the droplet is part of what is formally called and integral line. In what follows, we will use the concept of integral lines to define the stable and unstable manifolds of a function, and in turn use those definitions to define and describe the Morse-Smale decomposition.

1.1 Integral Lines

Formally, an integral line is a 'maximal path whose tangent vectors agree with the gradient' [1] at every point along the path. For the sake of brevity, we will henceforth denote by the term 'integral line' the image of said integral line. From this definition, some properties of interest arise:

1. An integral line is open on both ends;
2. The endpoints of an integral line are critical points of the manifold;
3. The set of integral lines covers the non-critical points of the manifold;
4. Integral lines do not intersect.

Let the source of integral line l be the maximal endpoint of l, denoted $source(l)$, and the let destination of l be the minimal endpoint of l, denoted $dest(l)$. We will use integral lines and their sources and destinations to define the stable and unstable manifolds of our surface M.

1.2 Stable and Unstable Manifolds

We define the stable and unstable manifolds with respect to a Morse function defined on a 2-manifold. Specifically, the critical points and integral lines to which we refer in this section (and its subsection) are the critical points and integral lines of said Morse function.

The stable manifold of a critical point p is defined as follows:
$S(p) = \{l | dest(l) = p\} \cup \{p\}$, where l is an integral line.
For a local maximum, its stable manifold is p; for a saddle, it's a pair of integral lines flowing into p from two maxima; for a local minimum, it's a 2-disk with p on its interior and a set of maxima and saddles (and the integral lines connecting them) on its boundary. The stable manifold of a local minimum p is also

sometimes called the $region$ of p. Returning to our example of the water droplet, the stable manifold of p is the set of points from which the droplet will flow into p.

The unstable manifold of a critical point p is defined as follows:
$U(p) = \{l | source(l) = p\} \cup \{p\}$.
The topology of the unstable manifold of p is similar to that of the stable manifold, but with the roles of the maxima and minima reversed: the unstable manifold of a maximum is a 2-disk; the unstable manifold of a minimum p is simply p. $U(p)$ is the set of paths that a water droplet placed at p may take.

1.2.1 Duality

Though it is of little relevance to Morse-Smale decomposition, it is interesting and perhaps useful for developing an intuition for the stable and unstable manifolds to see them as dual structures. Consider a local maximum p: $S(p)$ is a single point – a vertex. For a saddle, $S(p)$ is a pair of curves joined at p – an edge. For a minimum, $S(p)$ is a 2-disk – a face. By these definitions, the boundary of a face is a set of edges unioned with a set of vertices, and each edge is incident on exactly two vertices and two faces. Taking the dual of the stable manifold, the dual of a face should be a vertex, so the 2-disk formed by $S(p)$ has its dual as the local minimum p, which is $U(p)$; the dual of an edge $S(s)$ (where s is a saddle) connecting two maxima in the stable manifold is the edge $U(s)$ connecting two minima in the unstable manifold; the dual of a vertex (maximum) $S(q) = q$ in the stable manifold is the 2-disk (face) $U(q)$ about that maximum in the unstable manifold. We see that two adjacent faces (we call two two faces adjacent if they are incident on the same edge, and define it similarly for vertices) in one of these structures are dual to two adjacent vertices in the other structure; two adjacent vertices are dual to two adjacent faces; and the set of edges incident on a vertex (face, respectively) are dual to the set of edges incident on its dual face (dual vertex, respectively). One can thus see that by this duality that, given any critical point p, $U(p)$ and $S(p)$ are duals of one another.

2 Morse-Smale Decomposition

By overlaying the stable decomposition on the unstable decomposition, one obtains the Morse-Smale decomposition. Stated another way, the Morse-Smale complex is the collection of its Morse-Smale 'cells', and each 'cell' of the Morse-Smale complex is a non-empty intersection of the stable manifold of one critical point with the unstable manifold of another. Formally,
$MorseSmale(M) = \{U(p) \cap S(q) | p \text{ and } q \text{ are critical points of } M\}$.

2.1 Properties of Morse-Smale Cells

The above defnition leads to a couple of trivial but interesting properties of Morse-Smale cells:

1. All integral lines in a given Morse-Smale cell have the same source and destination.
2. Each k-cell of the Morse-Smale complex is homeomorphic to a k-dimensional open ball.

Furthermore, when M is a 2-manifold one may observe that each 1-cell lies between a saddle and and extreme point, each saddle point is incident on exactly four 1-cells, and each 2-cell has exactly one maximum, one minimum, and two saddles on its boundary, making it a quadrangle.

2.2 Dual

Consider the following construction for a dual to the Morse-Smale complex of a d-manifold: the dual of each k-cell is a $(d - k)$-face, with connectivity remaining the same (so, taking the 2-manifold example, the edges incident on a 2-cell f would be dual to edges incident on the vertex $dual(f)$). This dual complex is homotopic to M, which can be useful for computing the homology of M if M is initially expressed as a triangular mesh, as the dual to the Morse-Smale complex will often have far fewer simplices than an arbitrary triangulation of the surface and thus require significantly less computation.

2.3 Computing the Morse-Smale Complex

Computing the Morse-Smale complex of a piecewise-linear (PL) manifold is quite simple in theory; however, implementation of the algorithm we give here can be difficult. The algorithm has only two major steps:

1. Find the set of saddle points S of Morse function f defined on 2-manifold M.
2. From each $s \in S$ 'grow' four curves – two going up the (PL) gradient of f and two going down – until these curves reach local extrema.

Alternatively, for a simpler computation we can grow the curves in step 2 along edges of our PL-manifold, taking the edges of steepest ascent/descent. Finding saddles on a PL mesh has been discussed in a previous lecture, so we will not re-iterate the details here.

Using Discrete Morse Theory, there is a more combinatorial implementation; however, this is beyond the scope of this lecture.

3 Perturbation, Persistence, and Simplification

It is possible that the presence of some insignificant critical points give rise to large cells in the Morse-Smale complex, so one may be interested in performing some sort of computation to discover the significance of the critical points in the manifold and assign them some sort of persistence pairing. Given an extreme point p, take the set of saddles S adjacent to it in the Morse-Smale complex. Let $q = argmin_{s \in S}(|f(p) - f(s)|)$. p will cease to be an extreme point and q will cease to be a saddle if $f(p) := f(q)$, so this is what we do. p and q then form a persistence pairing and the significance of their persistence is equal to $|f(p) - f(q)|$. It is worth noting that if one were to carry out this procedure beginning with the saddle-extremum pair of least function difference (over all such saddle-extremum pairs on the manifold), and then add the pair with second-smallest function difference, and so forth such that each pair (p_i, q_i) added at iteration i satisfies $|f(p_i) - f(q_i)| < |f(p_{i+1}) - f(q_{i+1})| \forall i$, then the resulting pairing is the same as that obtained by the standard persistence algorithm.

The Morse-Smale complex can be maintained during the above persistence computation as follows. Consider two local maxima m_1 and m_2 such that $cls(U(m_1)) \cap cls(U(m_2))$ (where $cls(\sigma)$ is the closure of σ) is a curve c between two minima and passing through a saddle s that is the highest saddle adjacent to m_1 in the Morse-Smale complex. One may remove m_1 by lowering it to the level of s, forcing m_1 to no longer be a maximum and s to no longer be a saddle. The Morse-Smale complex must be updated to accomodate these changes to the manifold, and this update can be accomplished with relatively little computation as follows:

1. Find the set of integral lines L such that $l \in L$ has m_1 as its source in the original manifold M.
2. $\forall l \in L$, set $source(l) := m_2$ in the modified manifold M'.

Less formally, all cells of the Morse-Smale complex that are incident on m_1 of M are instead incident on m_2 in M'. Given the appropriate data stuctures, this makes for a very easy-to-implement combinatorial algorithm.

4 Acknowledgements

The majority of this discourse (with the exception of duality between the stable and unstable manifolds) is based on a lecture given by Prof. Yusu Wang, which drew heavily on the source cited below.

References

[1] Edelsbrunner, H., Harer, J., and Zomorodian, A. 2003. Hierarchical Morse-Smale complexes for piecewise linear 2-manifolds. Discrete Comput. Geom. 30, 87–107.