Lecture 11: Persistence induced by a function

Topics in Computational Topology: An Algorithmic View

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Today we will introduce filtration induced by Morse functions and how to compute the persistent pairing for critical points induced by Morse function.

1 More on Topology of Local Pattern

We will first summarize how the homology group of the subset level will change as it sweep through the critical points. With a little abuse of the notation, we call a small open neighborhood around a point p the star of the point p, which is introduced when we talked about simplical complex, with the topology of the open d-dimensional disk, $St(p) \approx R^d$. The closure of that is called the closed star of p, $\overline{St}(p) \approx B^d$. The link of p is the difference between the closed star and the star of p, $Lk(p) = \overline{St}(p) - St(p) \approx S^{d-1}$. To describe the local pattern, we need to talk about the portion within this small neighborhood with the function value that is lower than the point p. We call it lower star which is defined as $lowSt(p) = \{x \in St(p) \mid f(x) \leq f(p)\}$. Similarly, we can define lower link of p as $lowLk(p) = \{x \in Lk(p) \mid f(x) \leq f(p)\}$.

Theorem 1.1 *For a index s critical point, locally the function can be written as* $f(x) = -x_1^2 - x_2^2 - \ldots - x_s^2 + x_{s+1}^2 + \ldots + x_d^2 + f(p)$*, then lower link of p is homotopic to* S^{s-1}

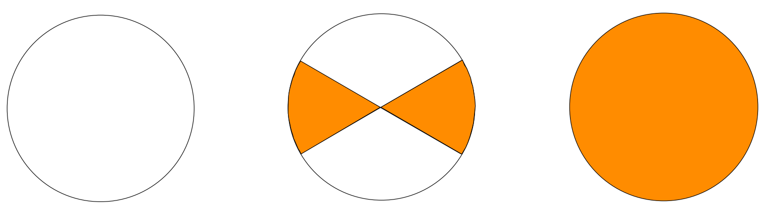

Figure 1: From left to right show the lower star of a minimum, a saddle and a maximum point in the 2D case

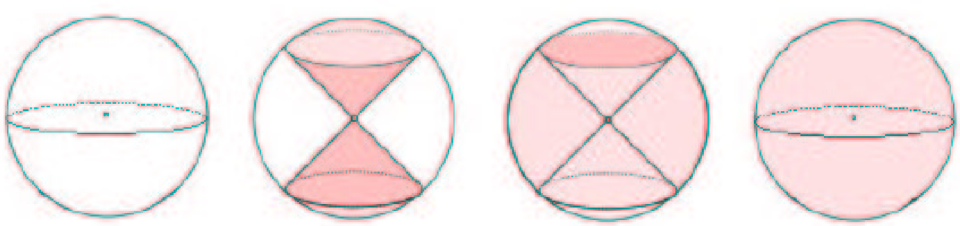

Figure 2: From left to right show the lower star of a index 0, 1, 2, 3 critical points in the 3D case (picture courtesy of [1]).

We already saw that when $s = 0$, it corresponds to minimum, and when $s = d$, it corresponds to maximum. In general, the tangent space of p is spanned by the d coordinates. Along the subspace spanned by s of them, the function value will be lower than $f(p)$, and the remaining will be higher than $f(p)$. Intuitively s out of d coordinates span the lower star of p, so $lowSt(p)$ has a dimension of s, a s dimensional disk. And the boundary of that, the $lowLk(p)$ is the $s-1$ dimensional sphere.

The lower star of every type of critical point in 2D case is shown in Figure 1, and 3D case is shown in Figure 2. This is a homological way to see what is around a critical point. And this has been summarized into the so called Handle Decomposition.

Theorem 1.2 *Let p be a index s critical point, $H_i(M^{(-\infty,f(p)]})$ is the same as the homology group of the space of gluing an s_handle on to $M^{(-\infty,f(p)-\varepsilon]}$*

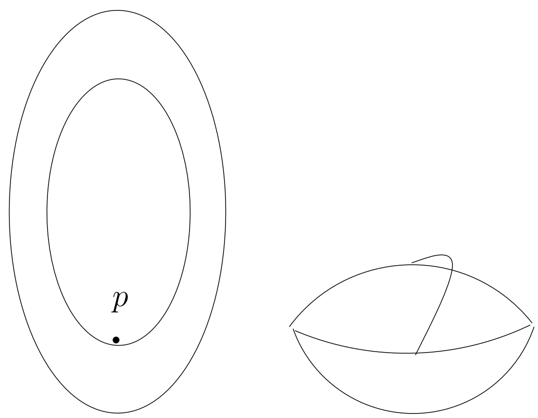

Figure 3: Left: the torus. Right: Sweep through the critical point p

Take the torus case in Figure 3 for example. When we pass through the critical point, it happens that things in the critical point p got pinched. The homology of this space is the same as when you keep the original space and glue a s handle on that. A s handle is simply a s dimensional disk intersecting with the original space. The boundary of the disk is the $s-1$ sphere. In Figure 3, The index of the critical point is 1, The homology of the sublevel set passing through p is equivalent to adding a one handle on it. One handle is a one dimensional disk, a segment. And we glue the boundary of the one dimensional disk to the original space. The boundary of the disk is two points, that means we add a segment to 2 points in the original space. Similarly, a two dimensional handle is a two dimensional disk, its boundary is a one dimensional sphere. Passing a index 2 critical point is the same as adding a two dimensional disk along a one dimensional cycle along the previous space.

What can the handle to do the the homology group? It turned out that an index s critical point will either increase β_s by 1 or decrease β_{s-1} by 1. There are only two possible cases. One is that the boundary of the s handle is already filled in the previously space, after gluing the disk, a void of s dimension will be formed, so β_s will increase by 1. In another case, the boundary is non-trivial before, when the handle is added, the non-trivial cycle is killed. In that case, β_{s-1} will decrease by 1.

2 Persistence induced by a function

2.1 Smooth Case

Suppose we have a morse function f, whose critical points are $p_1, p_2, \ldots, p_g$, and the corresponding critical value are $C_1, C_2, \ldots, C_g$, in increasing order. From the Morse Lemma, the homology group of the sublevel set will not change between any two consecutive critical points. Let $d_0, d_1, \ldots, d_g$ be a function value in between two consecutive critical value. The homology group between C_i and C_{i+1} will be the same as d_i. This forms a natural filtration.

$$\Phi = M^{(-\infty,d_0]} \subseteq M^{(-\infty,d_1]} \ldots \subseteq M^{(-\infty,d_g]} = M \tag{1}$$

The filtration naturally induce a inclusion map between these spaces, and inclusion map induces a homomorphism between their homology groups for any dimension.

$$H_i(M^{(-\infty,d_0]}) \to H_i(M^{(-\infty,d_1]}) \ldots \to H_i(M^{(-\infty,d_g]}) \tag{2}$$

Now we can define the persistent homology with respect to this filtration, the homomorphism is defined as

$$\rho^{i,j} : H(M^{(-\infty,d_i]}) \to H(M^{(-\infty,d_j]}) \tag{3}$$

The persistent Betti number is defined as

$$\beta^{i,j} = rank(Im(\rho^{i,j})) \tag{4}$$

and similarly as before,

$$\mu^{i,j} = (\beta^{i+1,j-1} - \beta^{i-1,j}) - (\beta^{i,j-1} - \beta^{i,j}) \tag{5}$$

if $\mu^{i,j} = k \neq 0$, There are k independent Homology classes created when passing through C_i and died when passing through C_j. This concept of persistence is very general, whenever you have a space induced by inclusion, you will get a map between homology groups and get the persistent homology induced by the filtration. When you have a Morse function, the homology group only changes when you pass through a critical point, so it suffices to consider the filtration when passing through every critical points and the persistent pairing between critical points.

2.2 Piecewise Linear Case

Previously we have a persistent algorithm, it only works for simplicial complex by adding one simplex each time. If we run the original algorithm, every simplex will either be a creator or a destroyer. You will get a pairing of every simplex instead of pairing of critical points. But here we need the pairing of critical points, what can we do?

Index	0	1	2
β_0	0	2	1
β_1	0	0	1
β_2	0	0	0

Table 1: Betti number of local pattern for 2D

Suppose we have a triangulation K whose underlying domain is the manifold $|K| \approx M$, A piecewise linear (pl-)function $f : |K| \to R$. Function values are given at vertices of K, and linearly interpolated within every simplex of K. As a result, maximum and minimum can only lie on the vertices. Next, we need to define critical points for the piecewise linear function. Remember in smooth case, we define critical points as where gradient vanishes. The gradient is constant within each face since it is linear in each face

Index	0	1	2	3
β_0	0	2	1	1
β_1	0	0	1	0
β_2	0	0	0	1
β_3	0	0	0	0

Table 2: Betti number of local pattern for 3D

and is well defined. But the gradient of vertex is not well defined. (If you take the largest gradient of its adjacent faces, then there is no vertex with vanishing gradient at all) Instead, we have to go back to the local pattern again.

For 2D case, we have the lower star as shown in the Figure 1 for the three type of critical points. The theorem says lower link is homotopic to $s-1$ dimensional sphere. The Betti number of the lower link is shown in Table 1. The lower star of 3D case is shown in Figure 2 and the Betti number Table 2.

For the s index critical points, its lower link is a $s-1$ sphere. β_{s-1} will be 1, everything else will 0 other than 0 dimensional case. For a d dimensional regular point, the lower link is a disk, all the Betti numbers are 0 except $\beta_0 = 1$. For mathematical elegance, we will define the so-called reduced Betti number, which is a fix for 0 dimension.

Definition 2.1 *Reduced Betti number:* $\tilde{\beta}_p = \beta_p$ *for* $p > 0$, $\tilde{\beta}_0 = \beta_0 - 1$ *and* $\tilde{\beta}_{-1} = 0$ *if* $\beta_0 \neq 0$ *and* $\tilde{\beta}_0 = 0$ *and* $\tilde{\beta}_{-1} = 1$ *if if* $\beta_0 = 0$

Now look at the reduced Betti number in Table 3 for 2D and Table 4 for 3D. We get a nice diagonal. For index s critical point, only the $s-1$ dimension reduced Betti number is 1, everyone else is 0. This gives the combinatorial way to decide the critical points on a mesh.

Index	0	1	2
$\tilde{\beta}_{-1}$	1	0	0
$\tilde{\beta}_0$	0	1	0
$\tilde{\beta}_1$	0	0	1
$\tilde{\beta}_2$	0	0	0

Table 3: Reduced Betti number of local pattern for 2D

Index	0	1	2	3
$\tilde{\beta}_{-1}$	1	0	0	0
$\tilde{\beta}_0$	0	1	0	0
$\tilde{\beta}_1$	0	0	1	0
$\tilde{\beta}_2$	0	0	0	1
$\tilde{\beta}_3$	0	0	0	0

Table 4: Reduced Betti number of local pattern for 3D

Definition 2.2 *A point P is an index k critical point iff $\tilde{\beta}_{k-1}(lowLk(p)) \neq 0$. It is a non-degenerate index k critical point if $\tilde{\beta}_{k-1} = 1$, and $\tilde{\beta}_i = 0, \forall i \geq 0$*

Now our mesh is K, its vertices is $v_1, v_2, \ldots, v_n$, in the increasing order of the function value. In the discrete case, every time you increase the function value such that it pass through another vertex. This give us the following filtration. $K_1 \subseteq K_2 \ldots \subseteq K_n$ where $K_i : \{\sigma | f(\sigma) \leq f(v_i)\}$. The difference between K_i and K_{i+1} is simply the $lowSt(v_{i+1})$.

In the matrix reduce, each time you just add one new simplex. It means every simplex must be ordered. Here, between K_i and K_{i+1}, there are simplices in $lowSt(v_{i+1})$, we still need to decide their order. It turned out that the order does not matter as long as the order maintain the validity of the simplicial complex. A simple way to add is to add by dimension. First to add 0 dimension simplex which is v_{i+1} and all the edges and all the faces. This guarantee that when you add a simplex, all its faces are already in.

Let's take a two manifold case for example. The first vertex v_1 is the global minimum. K_1 is v_1 itself. It does not paired with anything in its lower star. Now when we reach v_i, suppose it is a regular point (Figure 4 Left), its lower star forms a half disk. In the matrix reduction, you already have the matrix corresponding to K_{i-1}, when you add the new thing, the first column will be v_i, it creates a new component. Next, All the edges will be added, when adding the edge $v_{k1}v_i$, it destroyed the new component v_i and is a destroyer. Apparently, when you add all the remaining edges, it creates new non-trivial cycle and all are creators. Then all the triangles are added, they all destroyed a cycle created just now and gets paired with a edge. If you change the order, the pairing will change slightly, but every simplex will be paired with some simplex within the lower star.

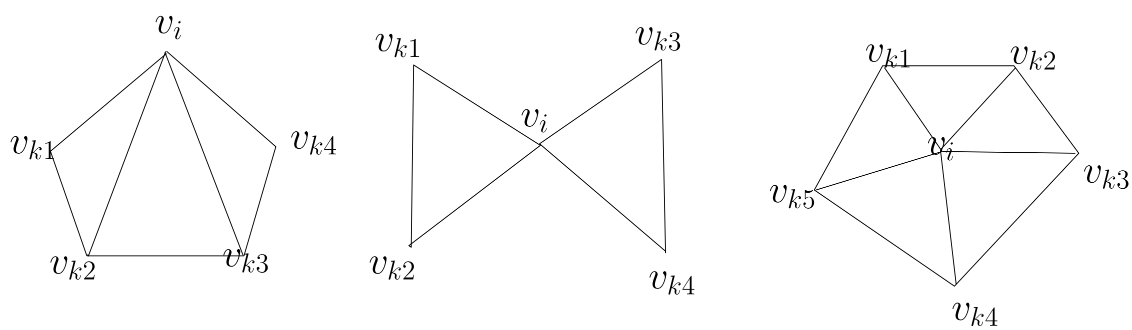

Figure 4: The lower star of a vertex. left: regular point. middle: saddle point. right: maximum point

However, if we meet a saddle point, the lower star will look Figure 4 Middle. It has two parts. Similarly, v_i is first added, it creates a new component. Next, the edges, $v_{k1}v_i$ destroyed the new component v_i and is a destroyer. $v_{k2}v_i$ creates a new cycle. Now for the edge $v_{k3}v_i$. It has two possibilities, it v_{k3} and v_i are already connected in K_{i-1}, then this edge creates a new cycle. If not, it destroys one connected component.

Suppose it is a destroyer, there must be a creator of one of the two component get killed. The point is that the paired creator is not in the lower star. If this edge is a creator, all the following edges are also creator. Then when you add the triangles, each of them killed one of the cycles in the lower star. But the big cycle created by $v_{k3}v_i$ not fully contained in the lower star is not killed. Its pairing will be in the future which is not in the lower star.

For a maximum, all its star will be its lower star. the vertex will be a creator. The first edge $v_{k1}v_i$ is a destroyer and be paired with v_i. All the other edges will be creator as well. Now we add the five triangles, the first four will kill a cycle created by the four creator edges and be paired with them. The last triangle will be unpaired in the lower star. It will actually kill a big cycle created before.

Let's summarize. For index 0 critical points, the vertex is not paired with anyone in the lower star. For index 1 critical point, there is an edge not paired within its lower star. For index 2 critical point, there is an triangle not paired within its lower star. For regular point, all the simplex got paired within the lower star. In general, If v_i is a index p critical points, if you add v_i and $lowSt(v_i)$ to K_{i-1}, then exactly one $p - simplex \in v_i \cup lowSt(v_i)$ is not paired within $lowSt(v_i)$. Eventually, you will have a persistent pairing of critical point if a simplex of v_i is paired with a simplex of v_j. Although the persistent algorithm will pair all the simplex. Most pairing will only be locally (within the lower star). By look at the pairing that is not in one lower star, you can get all the critical points and the persistent pairing.

Theorem 2.3 *Given a simplicial complex K whose underlying space is homeomorphic to a d-manifold M, and a PL-function $f : |K| \to \mathbb{R}$ defined on K, let $\{v_1, \ldots, v_n\}$ denote the set of vertices of K sorted in increasing function value. Using the filtration induced by adding each vertex and its lower-star in order, perform the standard persistence algorithm. A point v_i is an index-p critical if and only if there is one p-simplex in the lower-star of v_i that is either unpaired or its pairing-partner is not from the lower-star of v_i. There is a persistence pairing between an index p-critical point v_i and an index $(p+1)$-critical point v_j, if and only if there is one p-simplex from the lower-star of v_i is paired with an $(p+1)$-simplex from the lower-star of v_j.*

References

[1] H. Edelsbrunner and J. Harer. *Computational Topology: An Introduction*. 2009.