Chapter 1: Some Basics in Topology

Today we will introduce some basic concepts in topology. In this lecture, we will still stay in the continuous domain. After that, we will change gears to discrete setting, handling discrete objects (especially simplicial complexes) that we are more familiar with, and also that are more computationally friendly. References for the materials covered in this lecture include [2] (for Section 1 and 2) and [1] (for Section 3 and 4).

1 Topological Spaces

Definition 1.1 (Topological space) *A* topological space *is a set* X *endowed with a* topological structure *(a* topology*)* $\mathcal{T}$ *such that the following conditions are satisfied:*

1. *Both the empty set and* X *are elements of* $\mathcal{T}$*.*
2. *Any union of arbitrarily many elements of* $\mathcal{T}$ *is an element of* $\mathcal{T}$*.*
3. *Any intersection of finitely many elements of* $\mathcal{T}$ *is an element of* $\mathcal{T}$*.*

In other words, it is a set equipped with set of subsets. In particular, each set A in $\mathcal{T}$ is said to be *open*. Its complement $\mathsf{X} \setminus A$ is said to be *closed*. That is, a set A is *closed* if its complement is open. A set could be neither open nor closed. The *closure* of a set A is the minimal closed set from X containing A. Given the same space (set) X, one can defined different set system $\mathcal{T}$, which will then in turn lead to different topology. However, very often, in spaces (say Euclidean space, or the simplicial complex later) we encounter, we use the standard topology equipped with the space. Hence the reference to the set system $\mathcal{T}$ is often omitted.

This is a rather general and abstract definition. Now this looks a bit too alien and very abstract. Let us look some examples.

Example 1: Simple discrete topology on n **elemens.** Let X be a set of n elements $\mathsf{X} = \{x_1, \ldots, x_n\}$, and let $\mathcal{T} = 2^{\mathsf{X}}$. $< \mathsf{X}, \mathcal{T} >$ forms a topology. This is a *discrete* topology [1]. It is a simple topology. We can also consider the *trivial topology* on X, which is simply $\mathcal{T} = \{\emptyset, \mathsf{X}\}$.

Example 2: Metric topological space. Given a metric space (X, d_X), there is a natural way to put a topology on it. Let us now try to rephrase everything in the metric space. First, let us consider the Euclidean space $\mathsf{X} = \mathbb{R}^d$, with d_X being the standard Euclidean distance in $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$ and $\varepsilon > 0$, let $B(x, \varepsilon) := \{y \in \mathbb{R}^d \mid d(x, y) < \varepsilon\}$ be the open Euclidean ball around x, and let $\mathcal{B} = \{B(x, \varepsilon) \mid x \in \mathbb{R}^d, \varepsilon > 0\}$. Then set $\mathcal{T}$ to be the collection of (potentially infinite) union of any subset of open balls from $\mathcal{B}$, as well as the intersecition any finite subset of open balls from $\mathcal{B}$. (We also say that $\mathcal{B}$ is the basis that generates $\mathcal{T}$.) This gives the standard topology $(\mathsf{X} = \mathbb{R}^d, \mathcal{T})$ on Euclidean space $\mathbb{R}^d$.

Intuitively, $A \subset \mathbb{R}^d$ is open if at any point inside, we can move in arbitrary direction while still staying inside A. (Consider the more familiar *open intervals* on a line.)

[1] A topology $< \mathsf{X}, \mathcal{T} >$ is *discrete* if $\mathcal{T} = 2^{\mathsf{X}}$.

This definition of open set can be extended to any metric space[2] by replacing the distance $\|x - y\|$ with whatever metric distance $d(x, y)$ we need, which will give us a natural topology for any metric space.

The natural topology defined on a metric space perhaps is perhaps the most important and most common topological space.

Note that for the real line, the intersection of the intervals $(-\frac{1}{k}, +\frac{1}{k})$, for all integers $k \geq 1$, is the point 0. This is not an open set. This illustrates the need for the restriction to finite intersections.

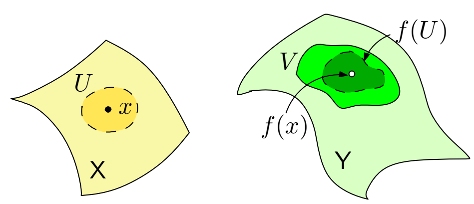
Subspace topology. Finally, suppose that we have a topological space $< \mathsf{X}, \mathcal{T} >$. Given a subset $\mathsf{Y} \subseteq \mathsf{X}$, it has a natural topology on it which is inherited from $\mathcal{T}$, denoted by $\mathcal{T}_Y$ defined as follows: open sets in $\mathcal{T}_Y$ is the intersection between open sets in $\mathcal{T}$ and Y. This is called the *subspace topology on* Y *induced from* $< \mathsf{X}, \mathcal{T} >$. For example, when we talk about topology for a surface S (or any compact set) embedded in $\mathbb{R}^3$, we in fact mean the subspace topology on S induced from $\mathbb{R}^3$. From now on, I will often omit the explicit reference of $\mathcal{T}$ and simply talk about a topological space X when the choice of $\mathcal{T}$ is clear. (In fact, we will mostly talk about the topology induced from a Euclidean space in this class.)

Remark. The topology (as well as the induced topology) in Euclidean space is the most common topological space one will encounter. The definition of topological space as sets of subsets may seem un-natural at first. In particular, the definitions of *open* and *closed* sets may be non-intuitive. Recall that we have said earlier that topology is about *connectivity* and about how the input space is put together from its subsets. Intuitively, this is captured by the open set and their union and intersections. Now when we compare two topological spaces, we need a map between them, and we need a language to say that the two spaces are connected in the same way using this map. The language for this purpose is *continuity*, which is one of the most important concepts in not just topology, but mathematics. We introduce it next.

2 Maps, homeomorphism, and homotopy

Definition 2.1 (Continuous function) *A* neighborhood *of a point $x \in \mathsf{X}$ is simply a subset of X that contains some open set U such that $x \in U$. A function $f : \mathsf{X} \to \mathsf{Y}$ is* continuous at $x \in \mathsf{X}$ *if for any neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$. See the right figure for an illustration. Function f is* continuous *if it is continuous at all points in X.*

Equivalently, a function $f : \mathsf{X} \to \mathsf{Y}$ is continuous *if for any open set V in Y, its preimage $f^{-1}(V)$ is also open.*



Note that the former is basically a generalization of the more familiar (ε, δ) definition of a continuous real-valued function on $\mathbb{R}$ from Calculus, that is, $f : \mathbb{R} \to \mathbb{R}$, which says that f is continuous at $x_0 \in \mathbb{R}$ if, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $x' \in (x_0 - \varepsilon, x_0 + \varepsilon)$ we have that $f(x') \in (f(x_0) - \delta, f(x_0) + \delta)$. (Consider the example where the function f has a jump at x.)

A continuous function $f : \mathsf{X} \to \mathsf{Y}$ between two topological spaces is also called a *map*.

[2]Recall a metric space is a space equipped with a distance $d(x, y)$ defined for any two elements $x, y \in \mathsf{X}$ such that the following conditions are satisfied: (1) $d(x, y) \geq 0$, (2) $d(x, y) = 0$ iff $x = y$, (3) $d(x, y) = d(y, x)$, (4) $d(x, y) \leq d(x, z) + d(z, y)$.

Homeomorphism. The most important concept to study topology is homeomorphism.

Definition 2.2 (Homeomorphism) *Given two topological spaces* X *and* Y*, a* homeomorphism *between them is a map* $h : \mathsf{X} \to \mathsf{Y}$ *such that* h *is bijection and the inverse of* h *is also continous.*

Two topological spaces are X *and* Y *are* homeomorphic*, denoted by* $\mathsf{X} \cong \mathsf{Y}$*, if there is a homeomorphism between them.*

Note that the inverse h^{-1} exists becaues h is bijective. The requirement that the inverse is also continuous is important. For example, look at the example $f : [0, 2\pi) \to \mathbb{S}_1$ with $f(x) = (\sin x, \cos x)$. f does not induce a homeomorphism between an interval and a circle as its inverse it not continuous.

Informally, bijection h means that there is a one-to-one correspondance between points in X and points in Y. That both f and its inverse are continuous mean that h further induces a one-to-one correspondance between open sets from X and open sets from Y, and all open sets from X are connected in the same way as their open sets in Y. Thus X and Y have the same topology.

Now let us look at a few examples.

(1) An open disk and $\mathbb{R}^2$. The explicit mapping is $f(x) = \frac{x}{1-\|x\|}$. (In fact, this map establishes a homeomorphism between the open d-ball and $\mathbb{R}^d$ for any $d > 0$.)

(2) Sphere and a tetrahedron. (From the center shoot a ray in all directions, and it intersects both tetrahedron and the sphere. That is the mapping f. (In fact, this map works for any two convex body of the same dimension.)

(3) Sphere with the north pole point removed and $\mathbb{R}^2$. (Again, shoot rays from north pole.)

We do not always need to find an explicit mapping to see that two spaces are homeomorphic. Intuitively, if one can deform from either one to the other without breaking and inserting, then they are homeomorphic. (Recall the examples we had from Lecture 0.)

Given a space X, we can consider its mapping into another target space.

Definition 2.3 *We say that* $\phi : \mathsf{X} \to \mathsf{Y}$ *is an* immersion *if locally for every point* $x \in \mathsf{X}$*, there is a small neighborhood* $U(x)$ *such that* Φ *induces an homeomorphism from* $U(x)$ *to* $\phi(U(x))$*.* ϕ *is an* embedding *if it induces a homeomorphism between* X *and* $\phi(\mathsf{X})$*.*

Homotopy equivalence. There is another notion of similarity among topological spaces that is weaker than homeomorphism, called homotopy equivalence. Intuitively, it relates spaces that can be continuously deformed to one another but may not be homeomorphic. For example, an annulus can continuously shrink to a circle, but they are not homeomorphic. To define homotopy equivalence, we hae to first define homotopy.

Definition 2.4 (Homotopy) *First, two* $f, g : \mathsf{X} \to \mathsf{Y}$ *are* homotopic *if there is a continuous map* $H : \mathsf{X} \times [0, 1] \to \mathsf{Y}$ *such that* $H_0 = f$ *and* $H_1 = g$*. This map* H *is called a* homotopy *connecting* f *and* g*.*

For example, if $f : \mathbb{B}^2 \to \mathbb{R}^2$ is the identify map, and $g : \mathbb{B}^2 \to \mathbb{R}^2$ maps every point in the disk to the origin, then f and g are homotopic, as established by the homotopy $H(x, t) := (1 - t) \cdot f(x)$; note $H(\mathbb{B}^2, t)$ deforms continuously from a disk at time 0 to a point at time 1.

We are now ready to use homotopies to define a relation between two spaces.

Definition 2.5 (Homotopy equivalence) *Two spaces* X *and* Y *are homotopy equivalent if there is a continuous mapping* $f : X \to Y$ *and* $g : Y \to X$ *such that* $f \circ g$ *is homotopic to identity in* Y *and* $g \circ f$ *is homotopic to identity in* X*.*

Show that $\mathbb{B}^2$ is homotopy equivalent to a single point in $\mathbb{R}^2$.

Theorem 2.6 *Two homeomorphic spaces are also homotopy equivalent, but not vice versa.*

In general, identifying homotopy equivalence via the above definition is not always easy. A specicial type of homotopy equivalence is *deformation retraction*, which is also one that rather common. This is more intuitive, compared to general homotopy equivalence.

Definition 2.7 (Deformation retraction) *Let $A \subseteq X$ be a subspace of topological space X. A* retraction (map) *r is a continuous map $r : X \to A$ such that $r(x) = x$ for any $x \in A$.*

We say that $A \subseteq X$ is a deformation retract of *X if there is a retraction r that is homotopic to the identity map in X. This retraction map is called a* deformation retraction.

Equivalently, a continuous map $R : X \times [0,1] \to X$ is a deformation retraction of *X* onto *A if for every $x \in X$ and $a \in A$, $R(x,0) = x$; $R(x,1) \in A$ and $R(a,1) = a$.*

As an example, an annulus and a circle, there is a natural retraction between them which is also a deformation retraction. There is a retraction from a ring to a point, but not a deformation retraction. In fact, any space can be retracted to a point. So a retraction does not preserve much topology. A deformation retract implies homotopy equivalence. There is no deformation retraction from a ring to a point.

If Y is a deformation retract of X, then X and Y are homotopy equivalient. In practice, another useful fact is that if two spaces are deformation retract of a common space, then they are homotopy equivalent. For example, ∞ (figure-8 shape) and $\emptyset$ are both deformation retract of a solid double torus (with thick neck). Hence they are homotopy equivalent.

Both homotopy and homeomorphsim are hard to test, especially in higher dimensions. From Lecture 3, we will start to talk about a more relaxed concept, called homology, which is meaningful, and at the same time can be easily computed.

Connectedness. A *path* is a continuous function from the unit interval, $\gamma : [0,1] \to \mathsf{X}$. A path is *closed* if $\gamma(0) = \gamma(1)$. A closed path is also called a *closed curve*. (An alternative way to define a close curve is a continuous function from the unit circle $\gamma : \mathbb{S}_1 \to \mathsf{X}$ (where $\mathbb{S}_1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$).) A path γ is *simple* if it is injective (other than potentially for $\gamma(0) = \gamma(1)$). A simple path does not have self-intersections. Often, we abuse the notation slightly and use a path to both refer to the map γ itself as well as its image $\gamma([0,1]) \subset \mathsf{X}$ in X.

Definition 2.8 ((Path-)Connected) *A topological space X is* path connected *if for every $x, y \in \mathsf{X}$, there is a path connecting them. A topological space X is* connected *if there does not exist two disjoint non-empty open sets $X_1, X_2 \subset \mathsf{X}$ such that $X_1 \cup X_2 = \mathsf{X}$. A maximal connected subset of X is called a* connected component *of X.*

The definition of connectedness is slightly weaker than path-connectedness. The following so-called $sin(1/x)$-space is connected but not path connected: Let $X = A \cup G \subset \mathbb{R}^2$ where $A = \{(0,y) \mid -1 \leq y \leq 1\}$ and $G = \{(x, sin(1/x)) \mid 0 < x \leq \pi/2\}$. It is connected because the component of X containing G is closed (components are always closed) and A is contained in the closure of G. But it is not path-connected.

However, for most spaces we will ever encounter, they are the same. Hence we limit ourselves to path-connectedness in this course.

3 Manifolds

First, some notations. $\mathbb{B}_d^o = \{x \in \mathbb{R}^d \mid \|x\| < 1\}$ is the open d-ball. We use $\mathbb{B}_d = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$ to denote the closed d-ball. The d-sphere is the boundary of a $d+1$ ball. That is, $\mathbb{S}_d = \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$.

What is $\mathbb{S}_0$, $\mathbb{S}_1$, etc? Recall that we have shown earlier that $\mathbb{B}_d^o \cong \mathbb{R}^d$; i.e, $\mathbb{B}_d^o$ is homeomorphic to $\mathbb{R}^d$. If we remove a single point from $\mathbb{S}_d$, what do we get? (We obtain a space homeomorphic to $\mathbb{B}_d^o$ and $\mathbb{R}^d$.)

Definition 3.1 (Manifold) *A d-*manifold (without boundary) *is a topological space* M *such that each point $x \in \mathsf{M}$ has a neighborhood homeomorphic to $\mathbb{R}^d$ (i.e, to $\mathbb{B}_d^o$).*

*A d-*manifold with boundary *is a topological space* M *such that each point $x \in \mathsf{M}$ either has a neighborhood homeomorphic to $\mathbb{R}^d$, or has a neighborhood homeomorphic to half-space $\mathbb{R}^d_+$.*

The collection of latter points are called the boundary *of the manifold.*

Theorem 3.2 (Boundary of A Manifold) *The boundary of a d-manifold is a $(d-1)$-dimensional manifold (possibly with multiple connected components).*

Basically, manifolds are those that locally it looks like the Euclidean space. Any closed curve is a 1-manifold. A curve is 1-manifold with boundary. A surface is 2-manifold, like a sphere, and a torus. In graphics, a solid is 3-manifold with boundary. Now some examples of non-manifolds: (1) a curve with branches, (2) two balls with a pinch point in the middle, (3) how about a sphere with a hole ? (this is manifold with boundary!) (4) Möbius strip (still manifold with boundary), (5) a book.

Embedding and immersion. Note that we define m-manifold in an abstract, local manner. A closed curve is a 1-manifold no matter which space it is in. Now we can embed an input m-manifold M in certain Euclidean space $\mathbb{R}^d$ (or any other metric space) via an embedding map $\Phi : \mathsf{M} \to \mathbb{R}^d$. The dimension m is called the *intrinsic dimension* of M. The space that M is embedded in is called the *ambient space* and its dimension is *ambient dimension*. For example, in our real life, we usually consider 1- or 2-manifolds embedded in $\mathbb{R}^3$. Note that ambient dimension is at least the intrinsic dimension.

4 2-manifolds and classification

2-manifolds, often referred to as *surfaces*, are of special interest, as they appear most often in real life, especially in graphics. The topological understanding of surfaces are quite thorough. In particular, it turns out that we can enumerate all kinds of surfaces with different topology in a simple manner. This is what we will first talk about: classification of surfaces.

A 2-manifold M is *compact* if for every covering of M by open sets, called an *open cover*, we can find a finite number of the sets that cover M. Examples of non-compact 2-manifolds include $\mathbb{R}^2$ itself, and any open subset of $\mathbb{R}^2$. We will focus on *compact surfaces* (with or without boundary) in what follows.

Orientability. First, consider the Möbius strip. Take a curve along the center, and walk a right-handed framework along it. After we make one circle, we reverse the orientation to left-handed. (Show them a paper example, and explain that after we finish a circle, we go to the other side of the same point, and this is equivalent to that the orientation of the framework changed (as the normal direction changed)). We call such a closed curve *orientation-reversing*; otherwise, it is *orientation-preserving*.

Definition 4.1 (Orientability) *A 2-manifold is* orientable *if all closed curves in it are orientation-preserving. Otherwise, it is non-orientable.*

It turns out that any compact surface without boundary in $\mathbb{R}^3$ is an orientable 2-manifold. A compact non-orientable 2-surface without boundary can only be embedded in a 4- or higher-dimensional Euclidean space. In $\mathbb{R}^3$, we can embed non-orientable surfaces *without boundary*. A Möbius strip is a 2-manifold with boundary that is non-orientable.

To obtain a non-orientable 2-manifold without boundary, simply glue the Möbius strip to a disk. This gives us the *projective plane* $\mathbb{P}$. Alternatively, if we remove a disk from the projective plane, we obtain the Mobius strip. Gluing the Mobius strip to the boundary of a disk is equivalent to gluing each pair of antipodal points from the boundary. To see this, consider the rectangular representation. We glue the top and bottom edges to the boundary of a disk (the circle). We can imagine that the top edge goes to $[0, \pi]$ half circle, and bottom edge goes to $[\pi, 2\pi]$. Now imagine that we shrink this rectangle to make its height goes to zero. Then the top and bottom edges will collapse. So a point in the top edge, say, corresponding to angle x, is identified to the point $\pi + x$ in the bottom one, which means that antipodal points are identified. In short, the projective plane is obtained by identifying antipodal points from boundary of a disk.

Note that we don't have to glue the Mobius strip to the boundary of a disk, it can be the boundary (or one boundary component if there are multiple ones) of any surface. Gluing two Mobius strips together, gives us the Klein bottle which is another well-known non-orientable surface (without boundary). Geometrically, imagine we glue two Mobius strips together along their boundaries, this is the same as we now get a cylindrical shape, where the two boundaries should be glued together but in the opposite orientation. Draw a picture (or bring a paper for example).

Connected sum. Now we introduce an operation that can be used to create new surfaces from known ones: the *connected sum* $M \# N$ of M and N is simply cutting a small disk from both surfaces, and then glue them via the boundaries. For example, $S \# T \approx T$, $T \# T \approx T^2$. If we perform connected sum of a surface and torus T, we call this *adding a handle*. If we perform connected sum of a surface with the projective plane $\mathbb{P}$, we call this *adding a cross cap*. (Ask 1: connected sum with $\mathbb{P}$ is the same as gluing the Mobius strip to the boundary circle, which is cross-identifying pairs of points. This is intuitively why we call this a cross cap.) (Ask 2: What is the connected sum between two projective planes? – Klein bottle, as this is simply gluing two Mobius strips together.)

Note that once we connected sum a surface with a projective plane, no matter what this surface is, and what furtuer connected sum operations we will perform, this surface stays non-orientable. The reason is because the equator of the Mobius strip that we glued on will always be there, which is a closed orientation-reversing closed cycle. The connected sum of a projective plane with a torus gives rise to a non-orientable surface which is homeomorphic to the connected sum of the projective plane and Klein bottle.

Classification of 2-manifolds. It turns out that among all possible surfaces, topologically, we have already seen all ingriedients of *compact surfaces*. Recall that a space X is *compact* if any open cover (namely, each set inside is open set) of it has finite subcover.

Theorem 4.2 (Classification Theorem) *The two infinite families $\mathbb{S}$, $\mathbb{T}$, $\mathbb{T}\#\mathbb{T}$, $\ldots$, and $\mathbb{P}$, $\mathbb{P}\#\mathbb{P}$, $\ldots$, exhaust the family of compact 2-manifold without boundary (upto homeomorphism). The first family of surfaces are all orientable; while the second family are all non-orientable. Furthermore, no two surfaces in these sequences are homeomorphic.*

In other words, a surface M is either homeomorphic to a sphere with g handles or to a sphere with g cross-caps to a sphere. In the former case, we call g the *genus* of the orientable surface M. Topologically, a surface is uniquely decided by this g-value and its orientability. g is a topological invariance. So if we are given two surfaces with different genus, then they cannot be homeomorphic.

Finally, to get a classification of compact 2-manifolds with boundary we can take one without boundary and make h holes by removing the same number of open disks. Each starting 2-manifold and each $h \geq 1$ give a different surface and they exhaust all possibilities. The number g, orientability, and number of holes, uniquely decide the manifold.

References

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[2] J. R. Munkres. *Topology*. Prentice Hall, Inc., 2 edition, 2000.