Local cliques in ER-perturbed random geometric graphs

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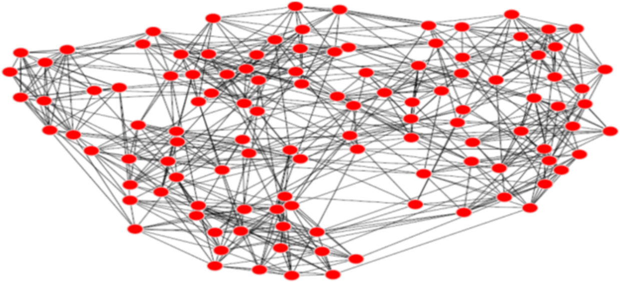
Joint work with Matthew Kahle and Yusu Wang

full version: https://arxiv.org/abs/1810.08383

November 6, 2018

Introduction

- Graph / Network — common data type



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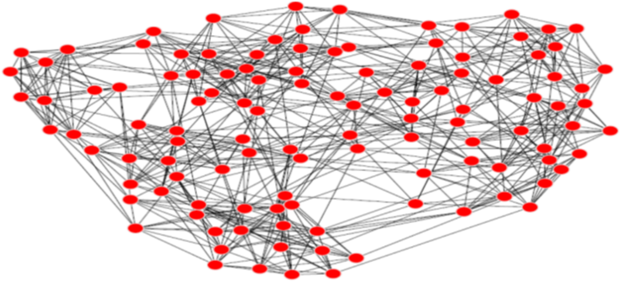

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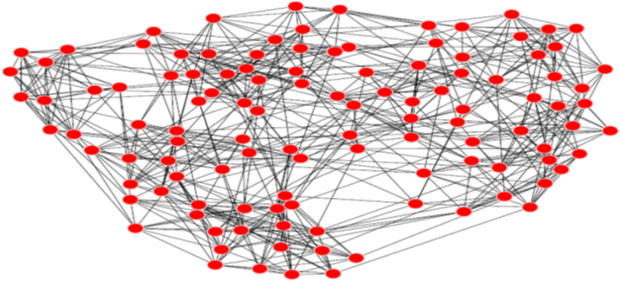
- High level goal:
  - Inference about true graph G^*, or analyze properties of perturbed graphs



Our Network Model

The true graph $G^* = (V, E^*)$ is a random geometric graph(RGG) where:

- $V = V_n$ are n points sampled *i.i.d* from a probability density function induced by a "nice" measure μ on a "nice" metric space $\mathcal{M} = (M, d)$

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- $p-$**deletion:** For each existing edge $(u, v) \in E^*$, we delete edge (u, v) with probability p
- $q-$**insertion:** For each non-existent edge $(u, v) \notin E^*$, we insert edge (u, v) with probability q

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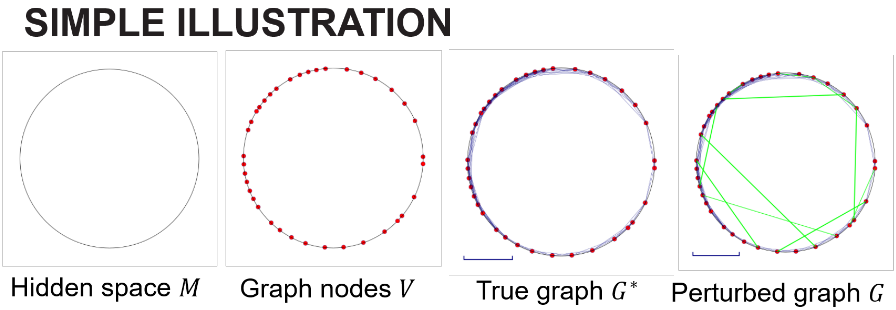

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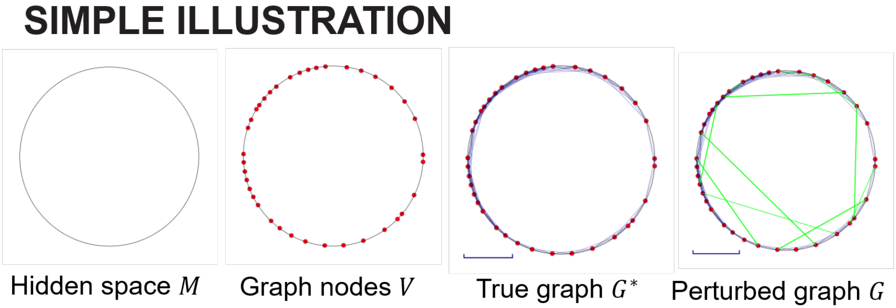

Remark: Our model is related to the continuum percolation theory[1].

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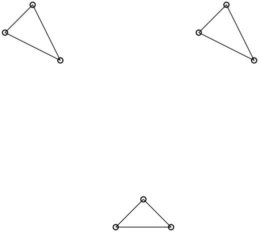
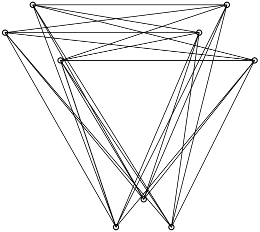
The Clique Number of The Union Graph



$\mathcal{G} = 3\ K_3$'s $(\omega(\mathcal{G}) = 3)$



The complement $\mathcal{G}^c$ $(\omega(\mathcal{G}^c) = 3)$

The union graph is a K_9 whose clique number is 9.
(consider $\sqrt{n}\ K_{\sqrt{n}}$ and its complement)

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- Intuitively, an edge in the observed graph G(ER-perturbed) can come from either the random geometric graph G^* or the Erdős-Rényi perturbation (inserted edges).
- (Main result) The edge clique number exhibits fundamentally different behaviors for these two types of edges.

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  - Every metric ball (with positive radius) has finite and positive measure and there is a constant $L = L(\mu)$ s.t. for all $x \in M$ and every $R > 0$, we have $\mu(B_{2R}(x)) \leq L \cdot \mu(B_R(x))$.

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  - L is the doubling constant and μ is an $L-$doubling measure.

Further Assumption on "Nice" Measure μ

For technical reasons, we need an assumption on the parameter r (for the RGG G^*), as well as a condition on the measure μ.

Assumption-A

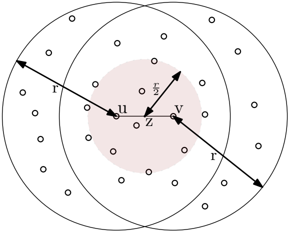
The parameter r and the doubling measure μ satisfy the following condition:

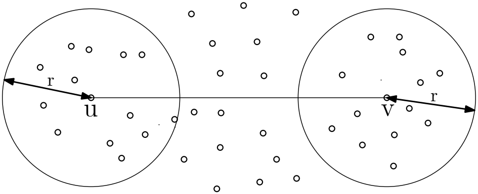
There exist $\mathrm{s} \geq \frac{13 \ln n}{n} \left(= \Omega(\frac{\ln n}{n})\right)$ and a constant ρ such that for any $x \in X$

(Density-cond) $\mu\left(B_{r/2}(x)\right) \geq \mathrm{s}$.

(Regularity-cond) $\mu\left(B_{r/2}(x)\right) \leq \rho \mathrm{s}$

Two Types of Edges — Good Edges and Bad Edges





Good edge: $d(u, v) \leq r$

(Edges from RGG)

Bad edge: $\forall x \in N_{G^*}(u), y \in N_{G^*}(v)$, $d(x, y) > r$.

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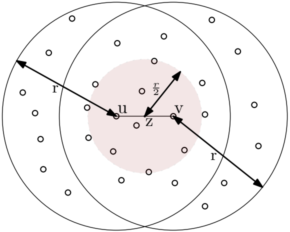

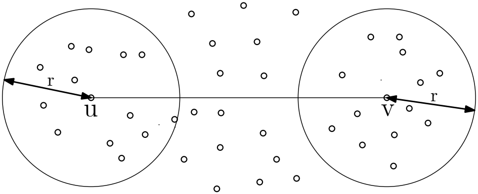

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Remark: There are "not-so-bad" edges other than these two types.

Main Results

Theorem (Simplified, Insertion-only)

Let G^* be the true graph generated as described, and G a graph obtained after random $q-$insertion. Under Assumption-A, for any insertion probability $q = o(1)$, with high probability,

- for all good edges $e \in G$, we have $\omega_G(e) = \Theta(sn)$
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Theorem (Simplified, $(p, q)-$perturbation)

Let G^* be the true graph generated as described, and G a graph obtained after random $p-$deletion and $q-$insertion. Under Assumption-A and assume $sn = \Theta(\ln n)$, for any constant $p \in (0, 1)$ and $q = o\left(\left(\frac{1}{n}\right)^{\frac{c}{\ln \ln n}} \frac{\ln \ln n}{\ln n}\right)$, with high probability,

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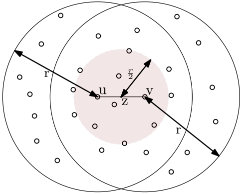
Sketch of Proof of the Insertion-only Case

- The lower bound for good edges



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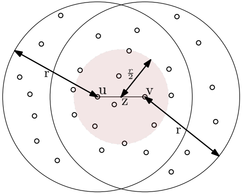

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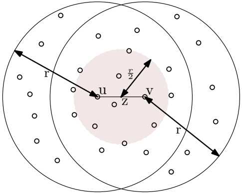

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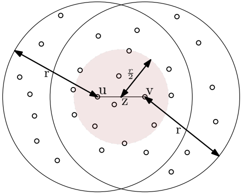

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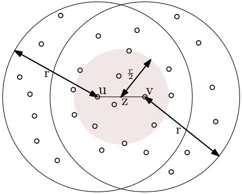

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  - Union bound

Sketch of Proof of the Insertion-only Case (cont'd)

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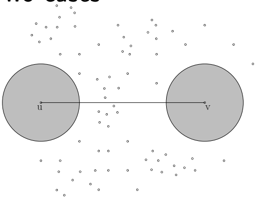
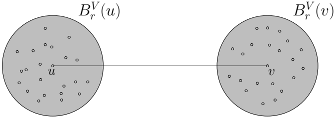


Case (a) — $\tilde{A}_{uv}$



Case (b) — B_{uv}

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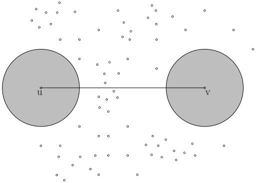
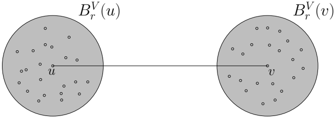


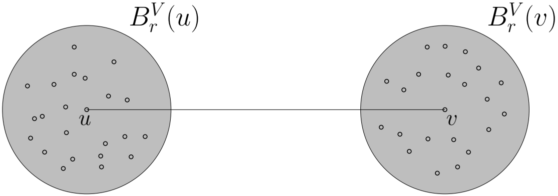


Case (a) — $\tilde{A}_{uv}$ Case (b) — B_{uv}

By the pigeonhole principle and the union bound, we have:

$$\mathbb{P}\left[G \text{ has a } uv\text{-clique of size} \geq \mathsf{K}\right]$$
$$\leq \mathbb{P}\left[G|_{\tilde{A}_{uv}} \text{ has a } uv\text{-clique of size} \geq \frac{\mathsf{K}}{2}\right] + \mathbb{P}\left[G|_{B_{uv}} \text{ has a } uv\text{-clique of size} \geq \frac{\mathsf{K}}{2}\right]$$

Case (b)



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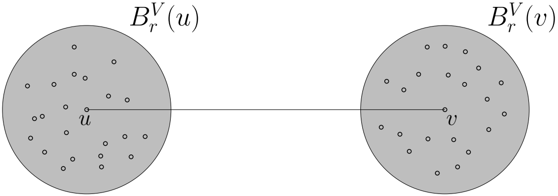

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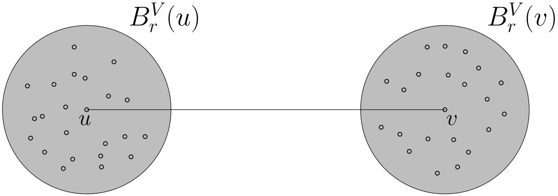

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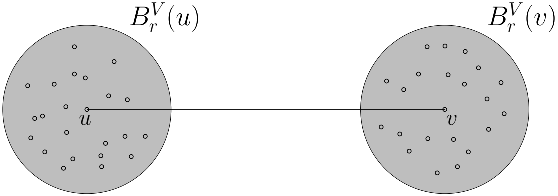

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- Estimate the expectation of the number of $uv-$cliques with size K/2
- Apply Markov's inequality (first moment method):
  $\mathbb{P}$[has a K/2 uv-clique] $\leq \mathbb{E}$[# of K/2 uv-cliques]

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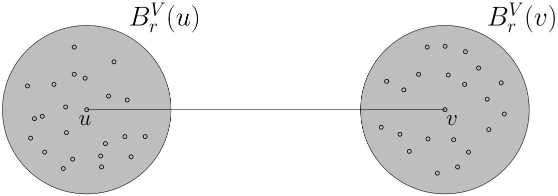

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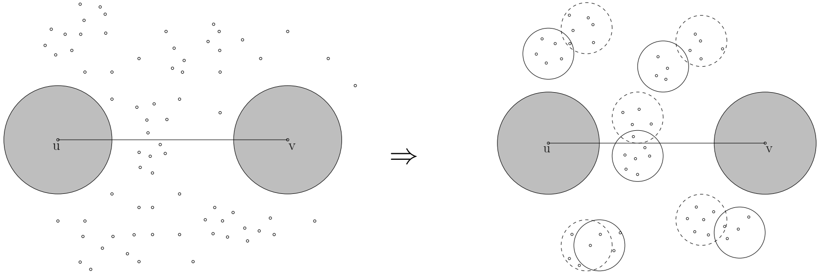
  $\mathbb{P}$[has a K/2 uv-clique] $\leq \mathbb{E}$[$\#$ of K/2 uv-cliques]
- To let the probability go to 0, we derive some requirement on q and K

Case (a)

- Decouple the randomness by checking **a constant number of** induced subgraphs (well-separated clique-partitions)



Well-separated Clique-partitions Family

Consider an arbitrary RGG $G^* = G^*_{\mathcal{X}}(r)$. A family $\mathcal{P} = \{P_i\}_{i \in \Lambda}$, where $P_i \subseteq V$ and Λ is the index set of P_is, forms a well-separated clique-partitions family of G^* if:

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2. $\forall i \in \Lambda$, P_i can be partitioned as $P_i = C_1^{(i)} \sqcup C_2^{(i)} \sqcup \cdots \sqcup C_{m_i}^{(i)}$ where

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2. $\forall i \in \Lambda$, P_i can be partitioned as $P_i = C_1^{(i)} \sqcup C_2^{(i)} \sqcup \cdots \sqcup C_{m_i}^{(i)}$ where
   (2-a) $\forall j \in [1, m_i]$, there exist $\bar{v}_j^{(i)} \in V$ such that $C_j^{(i)} \subseteq B_{r/2}\left(\bar{v}_j^{(i)}\right) \cap V$.

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We also call $C_1^{(i)} \sqcup C_2^{(i)} \sqcup \cdots \sqcup C_{m_i}^{(i)}$ a *clique-partition of* P_i *(w.r.t.* G^**)*, and its *size* (cardinality) is m_i. The *size* of the well-separated clique-partitions family $\mathcal{P}$ is its cardinality $|\mathcal{P}| = |\Lambda|$.

Well-separated Clique-partitions Family (cont'd)

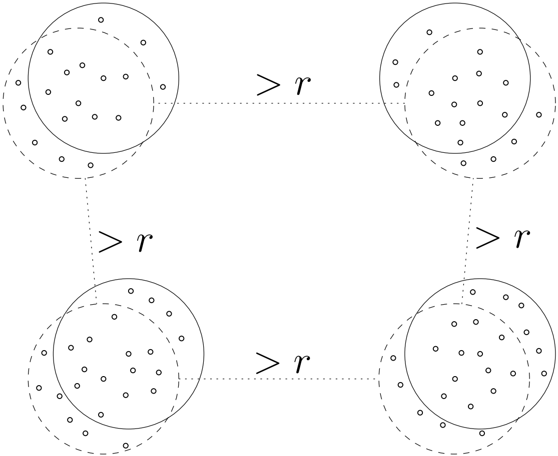

Figure: Points in the solid balls are P_1, and those in dashed balls are P_2. Each adapts a clique-partition of size $m_1 = m_2 = 4$. Assuming that all nodes in G^* are shown in this figure, then $\mathcal{P} = \{P_1, P_2\}$ forms a well-separated clique-partitions family of G^*.

Well-separated Clique-partitions Family (cont'd)

Theorem (Besicovitch Covering Lemma, doubling space version)

[a] *Let $\mathcal{X} = (X, d)$ be a doubling space. Then, there exists a constant $\beta = \beta(\mathcal{X}) \in \mathbb{N}$ such that for any $P \subset X$ and $\delta > 0$, there are β number of δ-packings w.r.t. P, denoted by $\{\mathcal{B}_1, \cdots, \mathcal{B}_\beta\}$, whose union also covers P.*

---

[a] Antti Kaenmaki, Tapio Rajala, and Ville Suomala. Local homogeneity and dimensions of measures. ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA-CLASSE DI SCIENZE, 16(4):1315 1351, 2016.

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We call the constant $\beta(\mathcal{X})$ above the *Besicovitch constant*.

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Theorem (Existence of finite-size W.S.C.P family)

Let $G^* = G^*_{\mathcal{X}}(r)$ be an n-node random geometric graph generated from $(\mathcal{X}, \mu, r)$ where $\mathcal{X} = (X, d)$ and μ is a doubling measure supported on X. There is a well-separated clique-partitions family $\mathcal{P} = \{P_i\}_{i \in \Lambda}$ of G^* with $|\Lambda| \leq \beta^2$, where $\beta = \beta(\mathcal{X})$ is the Besicovitch constant of $\mathcal{X}$.

Case (a) (cont'd)

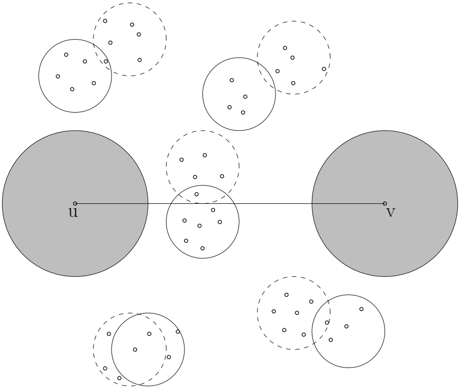
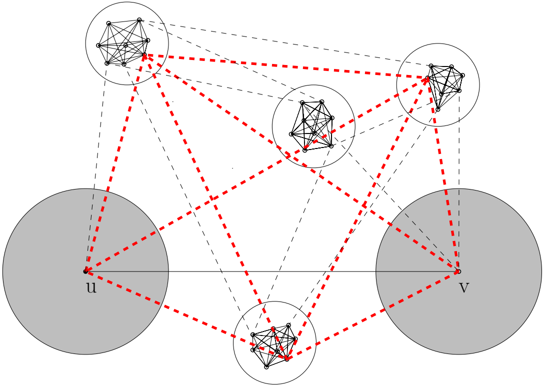


Figure: A well-separated clique partition $\mathcal{P} = \{P_1, P_2\}$ of A_{uv} — points in the solid ball are P_1, and those in dashed ball are P_2.

Case (a) (cont'd)

- In each induced subgraph (well-separated clique-partition)



Case (a) (cont'd)

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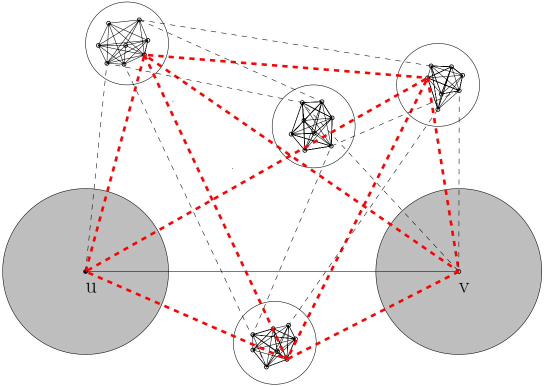


  - Use the same strategy as in case (b)

Clique Number in Standard RGG + $q-$Insertion

Theorem (By-product)

Suppose $n^{-\epsilon} \ll nr_n^d \ll \log n$ *for all* $\epsilon > 0$. *Then, for the* $q-$*perturbed random geometric graph* $\tilde{G}_q(\boldsymbol{X}_n; r_n)$, *the following holds*

- *If* $q \leq C_1 \left(\frac{nr_n^d}{\log n}\right)^{C_2}$, *where* C_1, C_2 *are two constants, then with high probability, we have*

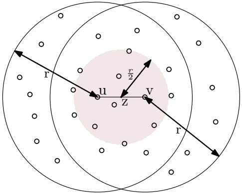
$$\omega\left(\tilde{G}_q(\boldsymbol{X}_n; r_n)\right) = \Theta\left(\frac{\log n}{\log \frac{\log n}{nr_n^d}}\right)$$

- *If* $\left(\frac{nr_n^d}{\log n}\right)^{\xi} \ll q \leq C_3$ *for all* $\xi > 0$ *where* C_3 *is a constant, then with high probability, we have*

$$\omega\left(\tilde{G}_q(\boldsymbol{X}_n; r_n)\right) = \Theta\left(\log_{\frac{1}{q}} n\right)$$

$(p, q)-$perturbation Case

- The lower bound for good edges



(p,q)−perturbation Case

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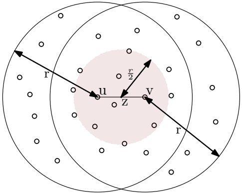

  - $B_r(u) \cap B_r(v)$ must contain an $r/2$ centered at the midpoint z of a geodesic connecting u to v in M

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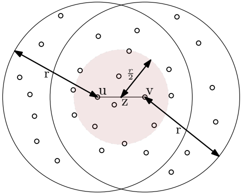
- The lower bound for good edges



  - $B_r(u) \cap B_r(v)$ must contain an $r/2$ centered at the midpoint z of a geodesic connecting u to v in M
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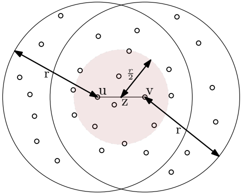

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  - The number of points in any $r/2-$ball can be bounded from below, say N_z, with **very** high probability by applying the Chernoff bound.

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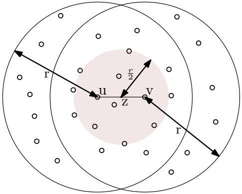

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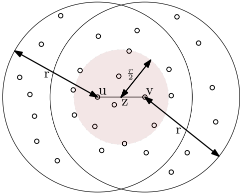

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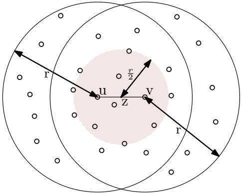

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$(p, q)-$perturbation Case

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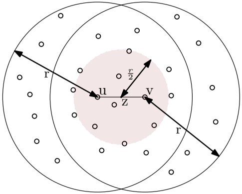

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  - Union bound
- The upper bound for bad edges (Same strategy!)

Clique Number in Standard RGG + $p-$Deletion

- For regime "$nr_n^d \leq n^{-\alpha}$ for some α"
  - Directly apply the Poisson approximation (the Stein-Chen method)
- For regime "$n^{-\epsilon} \ll nr_n^d \ll \log n$ for all $\epsilon > 0$"
  - Need put some constraint on p to fit the Poisson approximation setting
- For other regime? (e.g. $\frac{\sigma nr_n^d}{\log n} \to t \in (0, \infty)$)

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- Significantly larger range of insertion probability q than the case with Jaccard-filtering[4].

[4]Srinivasan Parthasarathy, David Sivakoff, Minghao Tian, and Yusu Wang. A quest to unravel the metric structure behind perturbed networks. In 33rd International Symposium on Computational Geometry, SoCG 2017, July 4-7, 2017, Brisbane, Australia, pages 53:153:16, 2017.

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Open Questions

- Other regimes? (e.g. sparse, thermodynamic limit, etc.)
- Other quantities to look at? (e.g. chromatic number, Lovász number)
- Other metric structures? (e.g. diffusion distance)

Thank you for your attention!