Computational Topology in Reconstruction, Mesh Generation, and Data Analysis

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Outline

- Topological concepts:
Outline

- Topological concepts:
  - Topological spaces
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  - Topological spaces
  - Maps

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Outline

- Topological concepts:
  - Topological spaces
  - Maps
  - Complexes
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  - Maps
  - Complexes
  - Homology groups

Applications:
- Manifold reconstruction
- Delaunay mesh generation
- Topological data analysis
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  - Delaunay mesh generation
  - Topological data analysis

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A point set with open subsets closed under union and finite intersections
Topology Background

**Topological spaces**

- A point set with **open** subsets closed under **union** and finite intersections

- $d$-ball $B^d \{ x \in \mathbb{R}^d \mid \|x\| \leq 1 \}$

- $d$-sphere $S^d \{ x \in \mathbb{R}^d \mid \|x\| = 1 \}$

- $k$-manifold: neighborhoods ‘homeomorphic’ to open $k$-ball
  - 2-sphere, torus, double torus are 2-manifolds

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Maps

- **Homeomorphism** $h : T_1 \rightarrow T_2$ where $h$ is continuous, bijective and has continuous inverse

- **Isotopy**: continuous deformation that maintains homeomorphism

- **Homotopy equivalence**: map linked to continuous deformation only
Simplicial complex

Abstract

- $V(K)$: vertex set, $k$-simplex: $(k + 1)$-subset $\sigma \subseteq V(K)$

Complex

$K = \{ \sigma \mid \sigma' \subseteq \sigma \implies \sigma' \in K \}$
Simplicial complex

- **Abstract**
  - $V(K)$: vertex set, $k$-simplex: $(k + 1)$-subset $\sigma \subseteq V(K)$
  - **Complex**
    - $K = \{ \sigma \mid \sigma' \subseteq \sigma \implies \sigma' \in K \}$

- **Geometric**
  - $k$-simplex: $k + 1$-point convex hull
  - **Complex $K$**:
    1. $t \in K$ if $t$ is a face of $t' \in K$
    2. $t_1, t_2 \in K \implies t_1 \cap t_2$ is a face of both

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Simplicial complex

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- **Triangulation**: $K$ is a triangulation of a topological space $T$ if $T \approx |K|$
Surface Reconstruction

Point Cloud

Surface Reconstruction

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Sampling

- Sample $P \subset \Sigma \subset \mathbb{R}^3$
Local Feature Size

\[ Lfs(x) \] is the distance to medial axis
ε-sample (Amenta-Bern-Eppstein 98)

Each $x$ has a sample within $\varepsilon Lfs(x)$ distance.
Crust and Cocone Guarantees

Theorem (Crust: Amenta-Bern 1999)

Any point $x \in \Sigma$ is within $O(\varepsilon)Lfs(x)$ distance from a point in the output. Conversely, any point of the output surface has a point $x \in \Sigma$ within $O(\varepsilon)Lfs(x)$ distance for $\varepsilon < 0.06$. 
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Theorem (Cocone: Amenta-Choi-Dey-Leekha 2000)

The output surface computed by COCONET from an $\varepsilon$-sample is homeomorphic to the sampled surface for $\varepsilon < 0.06$. 
Restricted Voronoi/Delaunay

Definition

Restricted Voronoi: $\text{Vor } P|_{\Sigma}$: Intersection of $\text{Vor } (P)$ with the surface/manifold $\Sigma$. 
Restricted Voronoi/Delaunay

Definition

Restricted Delaunay: $\text{Del } P|_{\Sigma}$: dual of $\text{Vor } P|_{\Sigma}$
Closed Ball property (Edelsbrunner, Shah 94)

If restricted Voronoi cell is a closed ball in each dimension, then Del $P|_{\Sigma}$ is homeomorphic to $\Sigma$. 
Closed Ball property (Edelsbrunner, Shah 94)

If restricted Voronoi cell is a closed ball in each dimension, then \( \text{Del } P|_{\Sigma} \) is homeomorphic to \( \Sigma \).
Topography

Closed Ball property (Edelsbrunner, Shah 94)

*If restricted Voronoi cell is a closed ball in each dimension, then Del $P|_{\Sigma}$ is homeomorphic to $\Sigma$.*

**Theorem**

*For a sufficiently small $\varepsilon$ if $P$ is an $\varepsilon$-sample of $\Sigma$, then $(P, \Sigma)$ satisfies the closed ball property, and hence Del $P|_{\Sigma} \approx \Sigma$.***
Closed Ball property (Edelsbrunner, Shah 94)

*If restricted Voronoi cell is a closed ball in each dimension, then Del $P|_\Sigma$ is homeomorphic to $\Sigma$.***
Closed Ball property (Edelsbrunner, Shah 94)

*If restricted Voronoi cell is a closed ball in each dimension, then $\text{Del } P|_\Sigma$ is homeomorphic to $\Sigma$.***
Boundaries

- Ambiguity in reconstruction
- Non-homeomorphic Restricted Delaunay [DLRW09]
- Non-orientability
Boundaries

- Ambiguity in reconstruction
- Non-homeomorphic Restricted Delaunay [DLRW09]
- Non-orientability

**Theorem (D.-Li-Ramos-Wenger 2009)**

*Given a sufficiently dense sample of a smooth compact surface $\Sigma$ with boundary one can compute a Delaunay mesh isotopic to $\Sigma$.***
Open: Reconstructing nonsmooth surfaces

- Guarantee of homeomorphism is open
High Dimensional PCD

- Curse of dimensionality (intrinsic vs. extrinsic)
High Dimensional PCD

- Curse of **dimensionality** (intrinsic vs. extrinsic)
- Reconstruction of submanifolds brings **ambiguity**
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High Dimensional PCD

- Curse of dimensionality (intrinsic vs. extrinsic)
- Reconstruction of submanifolds brings ambiguity
  - Use \((\varepsilon, \delta)\)-sampling
- Restricted Delaunay does not capture topology
  - Slivers are arbitrarily oriented [CDR05] \(\Rightarrow\) \(\text{Del } P|_{\Sigma} \not\approx \Sigma\) no matter how dense \(P\) is.
High Dimensional PCD

- Curse of *dimensionality* (intrinsic vs. extrinsic)
- Reconstruction of submanifolds brings *ambiguity*
  - Use $(\varepsilon, \delta)$-sampling
- Restricted Delaunay *does not* capture topology
  - Slivers are arbitrarily oriented [CDR05] \( \Rightarrow \) Del \( P|_{\Sigma} \not\cong \Sigma \) no matter how dense \( P \) is.
- Delaunay triangulation becomes *harder*
Reconstruction

Theorem (Cheng-Dey-Ramos 2005)

Given an \((\varepsilon, \delta)\)-sample \(P\) of a smooth manifold \(\Sigma \subset \mathbb{R}^d\) for appropriate \(\varepsilon, \delta > 0\), there is a weight assignment of \(P\) so that \(\text{Del} \hat{P}|_{\Sigma} \approx \Sigma\) which can be computed efficiently.
Reconstruction

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Theorem (Chazal-Lieutier 2006)

Given an \(\varepsilon\)-noisy sample \(P\) of manifold \(\Sigma \subset \mathbb{R}^d\), there exists \(r_p \leq \rho(\Sigma)\) for each \(p \in P\) so that the union of balls \(B(p, r_p)\) is homotopy equivalent to \(\Sigma\).
Reconstructing Compacts
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- Lfs vanishes, introduce $\mu$-reach and define $(\varepsilon, \mu)$-samples.
Reconstructing Compacts

- Lfs vanishes, introduce $\mu$-reach and define $(\varepsilon, \mu)$-samples.

**Theorem (Chazal-Cohen-S.-Lieutier 2006)**

*Given an $(\varepsilon, \mu)$-sample $P$ of a compact $K \subset \mathbb{R}^d$ for appropriate $\varepsilon, \mu > 0$, there is an $\alpha$ so that union of balls $B(p, \alpha)$ is homotopy equivalent to $K^\eta$ for arbitrarily small $\eta$.***
Surface and volume mesh
Surface and volume mesh
Delaunay refinement

- Pioneered by Chew89, Ruppert92, Shewchuk98
Delaunay refinement

- Pioneered by Chew89, Ruppert92, Shewchuk98
- To mesh some domain $D$

Initialize points $P \subset D$, compute Del$P$

If some condition is not satisfied, insert a point $p \in D$ into $P$ and repeat

Return Del$P | D$

Burden is to show termination (by packing argument)

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Delaunay refinement

- Pioneered by Chew89, Ruppert92, Shewchuk98
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Delaunay refinement

- Pioneered by Chew89, Ruppert92, Shewchuk98
- To mesh some domain $D$
  1. **Initialize** points $P \subset D$, compute $\text{Del } P$
  2. If some **condition** is not satisfied, **insert** a point $p \in D$ into $P$ and repeat
  3. Return $\text{Del } P|_D$
Delaunay refinement

- Pioneered by Chew89, Ruppert92, Shewchuk98
- To mesh some domain \( D \)
  1. Initialize points \( P \subset D \), compute \( \text{Del} \ P \)
  2. If some condition is not satisfied, insert a point \( p \in D \) into \( P \) and repeat
  3. Return \( \text{Del} \ P|_D \)
- Burden is to show termination (by packing argument)
Sampling Theorem

Theorem (Amenta-Bern 98, Cheng-D.-Edelsbrunner-Sullivan 01)

If $P \subset \Sigma$ is a discrete $\varepsilon$-sample of a smooth surface $\Sigma$, then for $\varepsilon < 0.09$, $\text{Del } P|_{\Sigma}$ satisfies:

- It is homeomorphic to $\Sigma$
- Each triangle has normal aligning within $O(\varepsilon)$ angle to the surface normals
- Hausdorff distance between $\Sigma$ and $\text{Del } P|_{\Sigma}$ is $O(\varepsilon^2)$
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- *It is homeomorphic to* \( \Sigma \)
- *Each triangle has normal aligning within* \( O(\varepsilon) \) *angle to the surface normals*
- *Hausdorff distance between* \( \Sigma \) *and* \( \text{Del} \ P|_{\Sigma} \) *is* \( O(\varepsilon^2) \) *of LFS.*
Sampling Theorem Modified

Theorem (Boissonnat-Oudot 05)

If $P \in \Sigma$ is such that each Voronoi edge-surface intersection $x$ lies within $\varepsilon \text{Lfs}(x)$ from a sample, then for $\varepsilon < 0.09$, $\text{Del } P|_{\Sigma}$ satisfies:

- It is homeomorphic to $\Sigma$
- Each triangle has normal aligning within $O(\varepsilon)$ angle to the surface normals
- Hausdorff distance between $\Sigma$ and $\text{Del } P|_{\Sigma}$ is $O(\varepsilon^2)$ of LFS.
Basic Delaunay Refinement

1. Initialize points $P \subset \Sigma$, compute $\text{Del } P$
2. If some \textbf{condition} is not satisfied, insert a point $c \in \Sigma$ into $P$ and repeat
3. Return $\text{Del } P|_{\Sigma}$
Surface Delaunay Refinement

1. Initialize points $P \subset \Sigma$, compute $\text{Del} \ P$

2. If some Voronoi edge intersects $\Sigma$ at $x$ with $d(x, P) > \varepsilon LFS(x)$, insert $x$ in $P$ and repeat

3. Return $\text{Del} \ P |_{\Sigma}$
How to compute $L_f(s(x))$?

Can be approximated by computing approximate medial axis—needs a dense sample.
Difficulty

- How to compute $L_f(s(x))$?
- Can be approximated by computing approximate medial axis—needs a dense sample.
A Solution

- Replace $d(x, P) < \varepsilon Lfs(x)$ with $d(x, P) < \lambda$, an user parameter.
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A Solution

1. Initialize points $P \subset \Sigma$, compute $\text{Del} \ P$

2. If some Voronoi edge intersects $\Sigma$ at $x$ with $d(x, P) > \lambda \text{LFS}(x)$, insert $x$ in $P$ and repeat

3. Return $\text{Del} \ P|_{\Sigma}$
A Solution

1. Initialize points $P \subset \Sigma$, compute $\text{Del} \ P$

2. If some Voronoi edge intersects $\Sigma$ at $x$ with $d(x, P) > \lambda \text{LFS}(x)$, insert $x$ in $P$ and repeat

3. If restricted triangles around a vertex $p$ do not form a topological disk, insert furthest $x$ where a dual Voronoi edge of a triangle around $p$ intersects $\Sigma$

4. Return $\text{Del} \ P|_{\Sigma}$
A Meshing Theorem

Theorem

Previous algorithm produces output mesh with the following guarantees:

1. Output mesh is always a 2-manifold.
2. If $\lambda$ is sufficiently small, the output mesh satisfies topological and geometric guarantees:
   - It is related to $\Sigma$ by an isotopy.
   - Each triangle has normal aligning within $O(\lambda)$ angle to the surface normals.
   - Hausdorff distance between $\Sigma$ and $\text{Del}P|\Sigma$ is $O(\lambda^2)$ of LFS.
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Data Analysis by Persistent Homology

- Persistent homology [Edelsbrunner-Letscher-Zomorodian 00], [Zomorodian-Carlsson 02]

Dey (2014) Computational Topology
Let $\mathcal{K}$ be a finite simplicial complex.
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**Definition**

A $p$-chain in $\mathcal{K}$ is a formal sum of $p$-simplices: $c = \sum_i a_i \sigma_i$; sum is the addition in a ring, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{R}$ etc.
Topological Data Analysis

**Chain**

- Let $\mathcal{K}$ be a finite simplicial complex

$$1\text{-chain } ab + bc + cd \ (a_i \in \mathbb{Z}_2)$$

**Definition**

A *p*-chain in $\mathcal{K}$ is a formal sum of $p$-simplices: $c = \sum_i a_i \sigma_i$; sum is the addition in a ring, $\mathbb{Z}, \mathbb{Z}_2, \mathbb{R}$ etc.
Boundary

Definition

A \( p \)-boundary \( \partial_{p+1}c \) of a \((p + 1)\)-chain \( c \) is defined as the sum of boundaries of its simplices.
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Boundary

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A \( p \)-boundary \( \partial_{p+1} \mathbf{c} \) of a \( (p + 1) \)-chain \( \mathbf{c} \) is defined as the sum of boundaries of its simplices.

2-chain \( bcd + bde \) (under \( \mathbb{Z}_2 \))
Boundary

Definition

A \textit{p-boundary} $\partial_{p+1}c$ of a $(p + 1)$-chain $c$ is defined as the sum of boundaries of its simplices.

1-boundary $bc + cd + db + bd + de + eb = \partial_2(bcd + bde)$ (under $\mathbb{Z}_2$)
Cycles

Definition

A \textit{p-cycle} is a \textit{p-chain} that has an empty boundary
Cycles

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![Simplicial complex](image)
Cycles

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A \( p \)-cycle is a \( p \)-chain that has an empty boundary.

\[ ab + bc + cd + de + ea \quad \text{(under } \mathbb{Z}_2) \]
Cycles

Definition

A \textit{p-cycle} is a \textit{p-chain} that has an empty boundary

1-cycle \( ab + bc + cd + de + ea \) (under \( \mathbb{Z}_2 \))

- Each \( p \)-boundary is a \( p \)-cycle: \( \partial_p \circ \partial_{p+1} = 0 \)
Groups

Definition

The $p$-chain group $C_p(K)$ of $K$ is formed by $p$-chains under addition.
Groups

Definition

The \textit{p-chain group} $\mathbb{C}_p(\mathcal{K})$ of $\mathcal{K}$ is formed by $p$-chains under addition.

The boundary operator $\partial_p$ induces a homomorphism

$$\partial_p : \mathbb{C}_p(\mathcal{K}) \to \mathbb{C}_{p-1}(\mathcal{K})$$
Groups

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Definition
The $p$-cycle group $Z_p(K)$ of $K$ is the kernel $\ker \partial_p$
Groups

Definition
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Definition
The \( p \)-cycle group \( Z_p(K) \) of \( K \) is the kernel \( \ker \partial_p \).

Definition
The \( p \)-boundary group \( B_p(K) \) of \( K \) is the image \( \text{im} \partial_{p+1} \).
Homology

Definition

The \( p \)-dimensional homology group is defined as

\[
H_p(\mathcal{K}) = Z_p(\mathcal{K}) / B_p(\mathcal{K})
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Homology

Definition
The \( p \)-dimensional homology group is defined as
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Definition
Two \( p \)-chains \( c \) and \( c' \) are homologous if
\[
c = c' + \partial_{p+1} d
\]
for some chain \( d \).
## Homology

**Definition**
The $p$-dimensional homology group is defined as
\[ H_p(K) = \frac{Z_p(K)}{B_p(K)} \]

**Definition**
Two $p$-chains $c$ and $c'$ are **homologous** if $c = c' + \partial_{p+1}d$ for some chain $d$

(a) trivial (null-homologous) cycle; (b), (c) nontrivial homologous cycles
Let $P \subset \mathbb{R}^d$ be a point set.
Complexes

- Let $P \subset \mathbb{R}^d$ be a point set
- $B(p, r)$ denotes an open $d$-ball centered at $p$ with radius $r$
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Definition

The \textit{Čech complex} $C^r(P)$ is a simplicial complex where a simplex $\sigma \in C^r(P)$ iff $\text{Vert}(\sigma) \subseteq P$ and $\bigcap_{p \in \text{Vert}(\sigma)} B(p, r/2) \neq \emptyset$
Complexes

- Let $P \subset \mathbb{R}^d$ be a point set
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**Definition**

The *Čech complex* $\mathcal{C}^r(P)$ is a simplicial complex where a simplex $\sigma \in \mathcal{C}^r(P)$ iff $\text{Vert}(\sigma) \subseteq P$ and $\bigcap_{p \in \text{Vert}(\sigma)} B(p, r/2) \neq 0$

**Definition**

The *Rips complex* $\mathcal{R}^r(P)$ is a simplicial complex where a simplex $\sigma \in \mathcal{R}^r(P)$ iff $\text{Vert}(\sigma)$ are within pairwise Euclidean distance of $r$
Complexes

Let $P \subset \mathbb{R}^d$ be a point set

$B(p, r)$ denotes an open $d$-ball centered at $p$ with radius $r$

**Definition**

The Čech complex $\mathcal{C}^r(P)$ is a simplicial complex where a simplex $\sigma \in \mathcal{C}^r(P)$ iff $\text{Vert}(\sigma) \subseteq P$ and $\bigcap_{p \in \text{Vert}(\sigma)} B(p, r/2) \neq 0$

**Definition**

The Rips complex $\mathcal{R}^r(P)$ is a simplicial complex where a simplex $\sigma \in \mathcal{R}^r(P)$ iff $\text{Vert}(\sigma)$ are within pairwise Euclidean distance of $r$

**Proposition**

For any finite set $P \subset \mathbb{R}^d$ and any $r \geq 0$, $\mathcal{C}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \mathcal{C}^{2r}(P)$
Point set $P$
Balls $B(p, r/2)$ for $p \in P$
Čech complex $C^r(P)$
Rips complex $\mathcal{R}^r(P)$
Topological persistence

- $r(x) = d(x, P)$: distance to point cloud $P$
- **Sublevel sets** $r^{-1}[0, a]$ are union of balls
- Evolution of the sublevel sets with increasing $a$—left hole persists longer
- Persistent homology quantizes this idea

![Diagram](image_url)
Persistent Homology

- \( f : \mathbb{T} \rightarrow \mathbb{R}; \ T_a = f^{-1}(-\infty, a] \), the sublevel set
- \( T_a \subseteq T_b \) for \( a \leq b \) provides inclusion map \( \iota : T_a \rightarrow T_b \)
- Induced map \( \iota_* : H_p(T_a) \rightarrow H_p(T_b) \) giving the sequence

\[
0 \rightarrow H_p(T_{a_1}) \rightarrow H_p(T_{a_2}) \rightarrow \cdots \rightarrow H_p(T_{a_n}) \rightarrow H_p(\mathbb{T})
\]

- Persistent homology classes: Image of \( f_{ij} : H_p(T_{a_i}) \rightarrow H_p(T_{a_j}) \)
Continuous to Discrete

- Replace $\mathbb{T}$ with a simplicial complex $K := K(\mathbb{T})$
Continuous to Discrete

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- Union of balls with its nerve Čech complex
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(a) (b)
(c) (d)
Continuous to Discrete

- Replace $\mathbb{T}$ with a simplicial complex $K := K(\mathbb{T})$
- Union of balls with its nerve Čech complex
- Evolution of sublevel sets becomes Filtration:

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$
$$0 \to H_p(K_1) \to \cdots \to H_p(K_n) = H_p(K).$$
Continuous to Discrete

- Replace $\mathbb{T}$ with a simplicial complex $K := K(\mathbb{T})$
- Union of balls with its nerve Čech complex
- Evolution of sublevel sets becomes Filtration:

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- Birth and Death of homology classes
Bar Codes

- birth-death and bar codes
Persistence Diagram

- a bar $[a, b]$ is represented as a point in the plane
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Stability of Persistence Diagram

- Bottleneck distance (\( C \) all bijections)

\[
d_B(\text{Dgm}_p(f), \text{Dgm}_p(g)) := \inf_{c \in C} \sup_{x \in \text{Dgm}_p(f)} \| x - c(x) \|
\]

Theorem (Cohen-Steiner, Edelsbrunner, Harer 06)

\[
d_B(\text{Dgm}_p(f), \text{Dgm}_p(g)) \leq \| f - g \|_{\infty}
\]
Back to Point Data

- $d_T$ be the distance function from the space $T$. 
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- Efficient algorithm for Zigzag simplicial maps?
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- Compute an optimal set of cycles forming a basis
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H$_1$ basis for simplicial complexes: D.-Sun-Wang [SoCG10]
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- Special cases: Dey-Hirani-Krishnamoorthy [STOC10]
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  - connecting to data mining, machine learning.
Thank You