Topological Analysis of Nerves, Reeb Spaces, Mappers, and Multiscale Mappers

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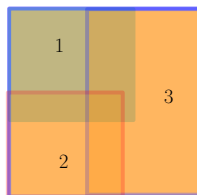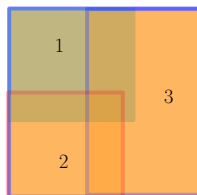
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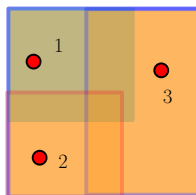
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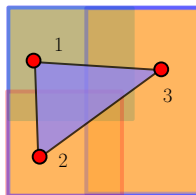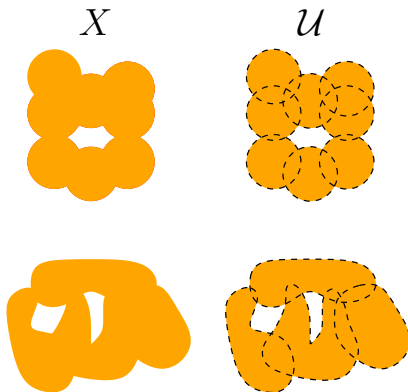
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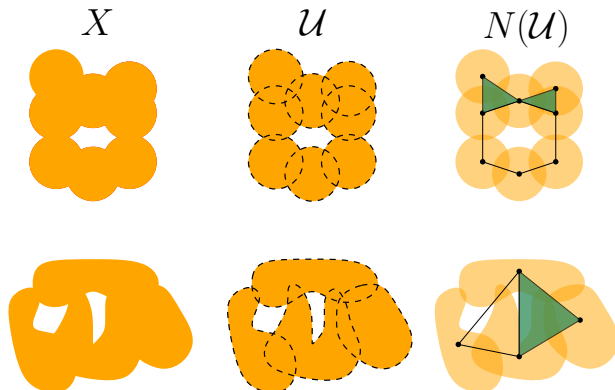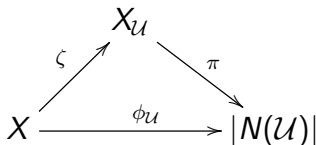
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From space to nerve and H_1-classes

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- $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, a path connected cover, $X_{\mathcal{U}}$: blowup space
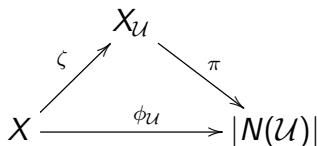- $\phi_{\mathcal{U}} : X \to |N(\mathcal{U})|$ is a map where $\phi_{\mathcal{U}} = \pi \circ \zeta$

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$$
\begin{array}{ccc}
 & X_{\mathcal{U}} & \\
\zeta \nearrow & & \searrow \pi \\
X \xrightarrow{\quad \phi_{\mathcal{U}} \quad} & & |N(\mathcal{U})|
\end{array}
$$

Theorem (Space-Nerve)

$\phi_{\mathcal{U}*} : H_1(X) \to H_1(|N(\mathcal{U})|)$ is a surjection.

Maps between covers

- Consider covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ and a map of sets $\xi : A \to B$ satisfying $U_\alpha \subseteq V_{\xi(\alpha)}$ for all $\alpha \in A$

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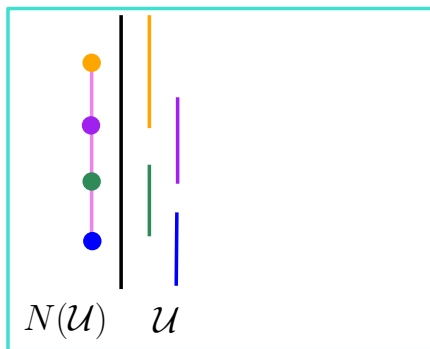- ξ induces a simplicial map $N(\xi) : N(\mathcal{U}) \to N(\mathcal{V})$

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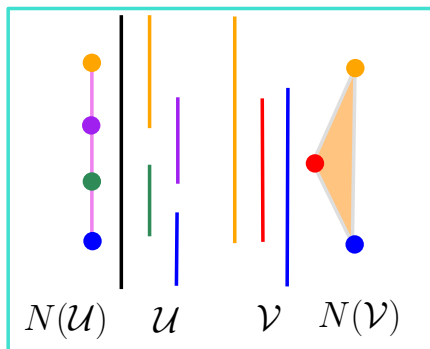- ξ induces a simplicial map $N(\xi) : N(\mathcal{U}) \to N(\mathcal{V})$

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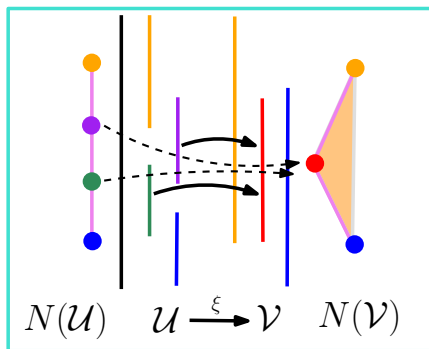- ξ induces a simplicial map $N(\xi) : N(\mathcal{U}) \to N(\mathcal{V})$

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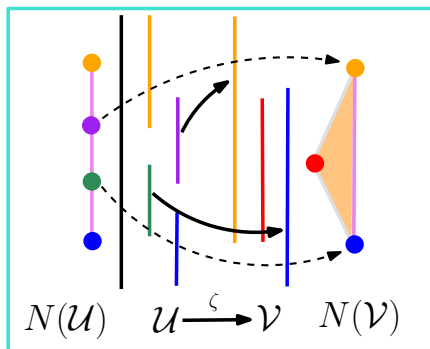- ξ induces a simplicial map $N(\xi) : N(\mathcal{U}) \to N(\mathcal{V})$
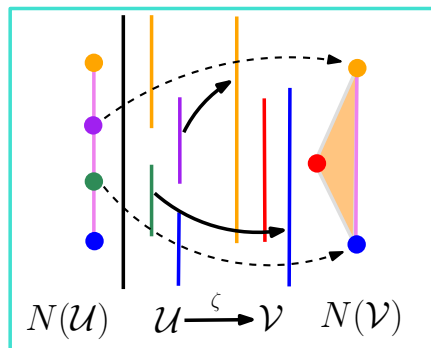
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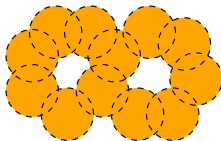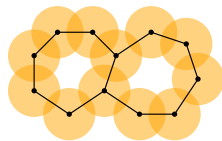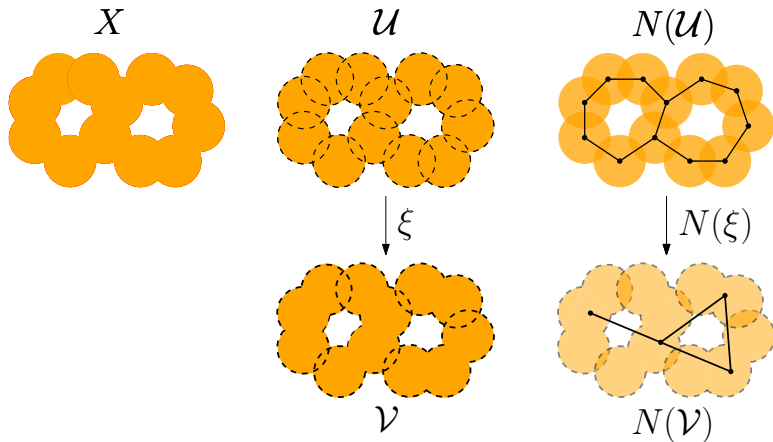- if $\mathcal{U} \xrightarrow{\xi_1} \mathcal{V} \xrightarrow{\xi_2} \mathcal{W}$, then $N(\xi_2 \circ \xi_1) = N(\xi_2) \circ N(\xi_1)$

Nerve to nerve and H_1-classes

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Proposition

$\mathcal{U}$ and $\mathcal{V}$ be two covers of X with a cover map $\mathcal{U} \xrightarrow{\theta} \mathcal{V}$. Then, $\phi_{\mathcal{V}} = \hat{\tau} \circ \phi_{\mathcal{U}}$ where $\tau : N(\mathcal{U}) \to N(\mathcal{V})$ is induced by θ.

Corollary

The maps $\phi_{\mathcal{U}*} : H_k(X) \to H_k(|N(\mathcal{U})|)$, $\phi_{\mathcal{V}*} : H_k(X) \to H_k(|N(\mathcal{V})|)$, and $\hat{\tau}_* : H_k(|N(\mathcal{U})|) \to H_k(|N(\mathcal{V})|)$ commute, that is, $\phi_{\mathcal{V}*} = \hat{\tau}_* \circ \phi_{\mathcal{U}*}$.

Theorem (Nerve-Nerve)

Let $\tau : N(\mathcal{U}) \to N(\mathcal{V})$ be induced by a cover map $\mathcal{U} \to \mathcal{V}$. Then, $\tau_* : H_1(N(\mathcal{U})) \to H_1(N(\mathcal{V}))$ is a surjection.

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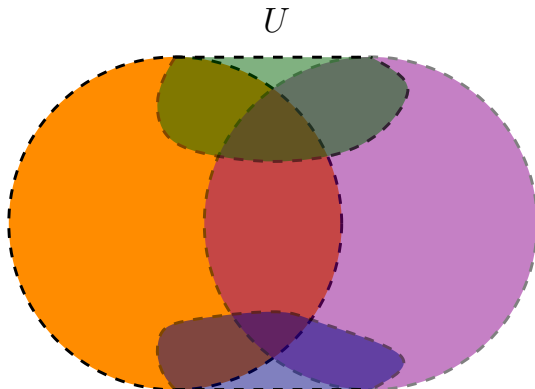# Lebesgue number of a cover

Lebesgue number of a cover:

$$\lambda(\mathcal{U}) = \sup\{\delta \mid \forall X' \subseteq X \text{ with } s(X') \leq \delta, \exists U_\alpha \in \mathcal{U} \text{ where } U_\alpha \supseteq X'\}$$

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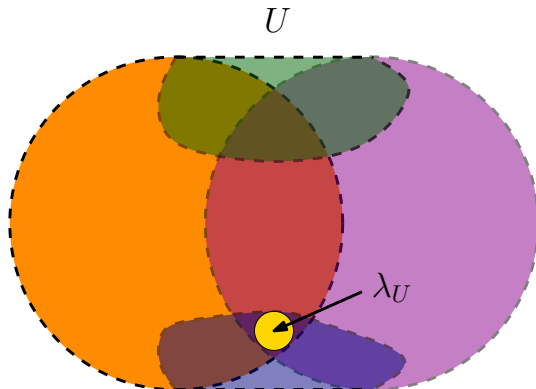# Lebesgue number of a cover

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Persistent H_1-classes

Theorem (Persistent H_1-classes)

Let $z_1, z_2, \ldots, z_g$ be a minimal generator basis of $H_1(X)$ ordered with increasing sizes.

   i. *Let $\ell \in [1, g]$ be the smallest integer so that $s(z_\ell) > \lambda(\mathcal{U})$. If $\ell \neq 1$, the class $\bar{\phi}_{\mathcal{U}*}[z_j] = 0$ for $j = 1, \ldots, \ell - 1$. Moreover, the classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell,\ldots,g}$ generate $H_1(N(\mathcal{U}))$.*

   ii. *The classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell',\ldots,g}$ are linearly independent where $s(z_{\ell'}) > 4s_{max}(\mathcal{U})$.*

Reeb space

- Reeb graphs capture only vertical homology classes [D.-Wang 14]
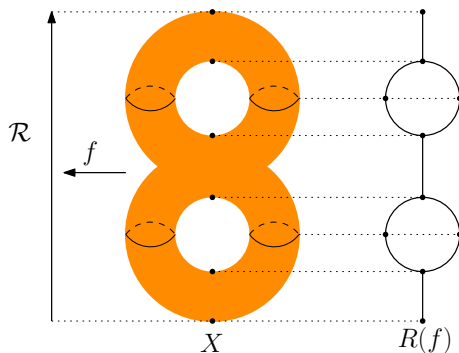
Theorem (Reeb space)

Only vertical H_1-classes survive in Reeb spaces.

Maps and covers
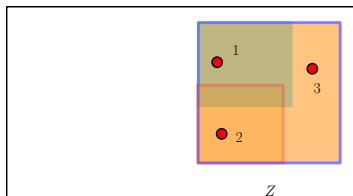
- Let $f : X \rightarrow Z$ continuous, well-behaved and $\mathcal{U}$ a finite cover of Z.

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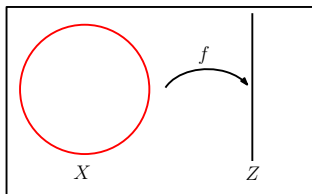

Definition (Mapper)

[Singh-Carlsson-Mémoli] Let $f : X \to Z$ be continuous and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a finite open covering of Z. The Mapper is

$$\mathrm{M}(\mathcal{U}, f) := N(f^*(\mathcal{U}))$$

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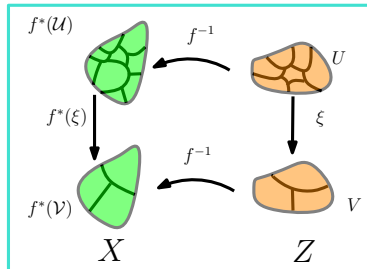
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- if $\mathcal{U} \xrightarrow{\xi} \mathcal{V} \xrightarrow{\zeta} \mathcal{W}$, then $f^*(\zeta \circ \xi) = f^*(\zeta) \circ f^*(\xi)$

Dey,Mémoli,Wang (2017)   Topological Analysis of Nerves

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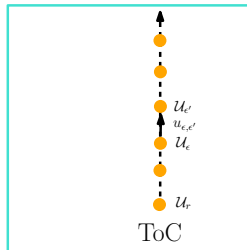 - $\mathfrak{S} = \left\{ \mathcal{S}_{\varepsilon} \xrightarrow{s_{\varepsilon, \varepsilon'}} \mathcal{S}_{\varepsilon'} \right\}_{r \leq \varepsilon \leq \varepsilon'}$, $s_{\varepsilon, \varepsilon} = \mathrm{id}$, $s_{\varepsilon', \varepsilon''} \circ s_{\varepsilon, \varepsilon'} = s_{\varepsilon, \varepsilon''}$

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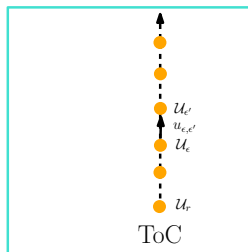 - $\mathfrak{U} = \left\{ \mathcal{U}_\varepsilon \xrightarrow{u_{\varepsilon,\varepsilon'}} \mathcal{U}_{\varepsilon'} \right\}_{r \leq \varepsilon \leq \varepsilon'}$, $u_{\varepsilon,\varepsilon} = \mathrm{id}$, $u_{\varepsilon',\varepsilon''} \circ u_{\varepsilon,\varepsilon'} = u_{\varepsilon,\varepsilon''}$

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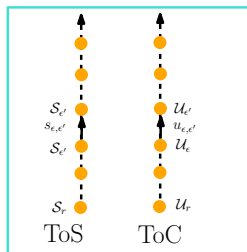
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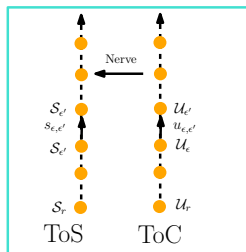

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Multiscale Mapper

- $f : X \to Z$ continuous, well-behaved, $\mathfrak{U}=$ ToC of Z
- Then, $f^*(\mathfrak{U})$ is ToC of X and $N(f^*(\mathfrak{U}))$is ToS

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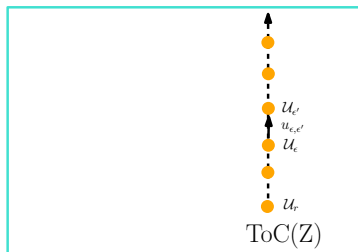
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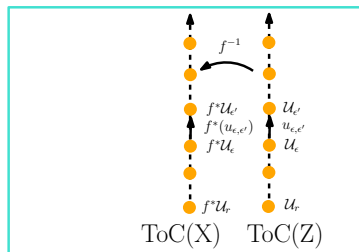- Then, $f^*(\mathfrak{U})$ is ToC of X and $N(f^*(\mathfrak{U}))$is ToS
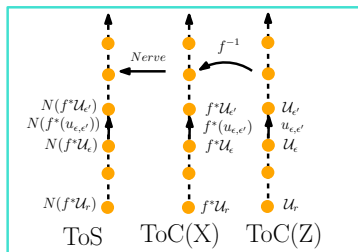
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Multiscale Mapper:

$$\mathrm{MM}(\mathfrak{U}, f) := N(f^*(\mathfrak{U}))$$

Persistence diagram of MM

- $D_k MM(\mathfrak{U}, f)=$ persistence diagram of:

$$H_k\big(N(f^*(\mathcal{U}_{\varepsilon_1}))\big) \to H_k\big(N(f^*(\mathcal{U}_{\varepsilon_2}))\big) \to \cdots \to H_k\big(N(f^*(\mathcal{U}_{\varepsilon_n}))\big)$$

Implication for multiscale mapper

Theorem

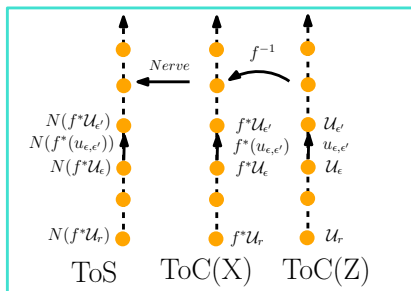*Consider the following multiscale mapper:*

$$N(f^*\mathcal{U}_0) \to N(f^*\mathcal{U}_1) \to \cdots \to N(f^*\mathcal{U}_n)$$

- *surjection from $H_1(X)$ to $H_1(N(f^*\mathcal{U}_i))$ for each $i \in [0, n]$.*
- *For H_1-persistence module:*

$$\mathrm{H}_1\big(N(f^*\mathcal{U}_0)\big) \to \mathrm{H}_1\big(N(f^*\mathcal{U}_1)\big) \to \cdots \to \mathrm{H}_1\big(N(f^*\mathcal{U}_n)\big)$$

all connecting maps are surjections.

Persistent H_1-classes in multiscale mapper

- $f : X \to Z$ where (Z, d_Z) a metric space
- $d_f(x, x') := \inf_{\gamma \in \Gamma_X(x, x')} \operatorname{diam}_Z(f \circ \gamma).$

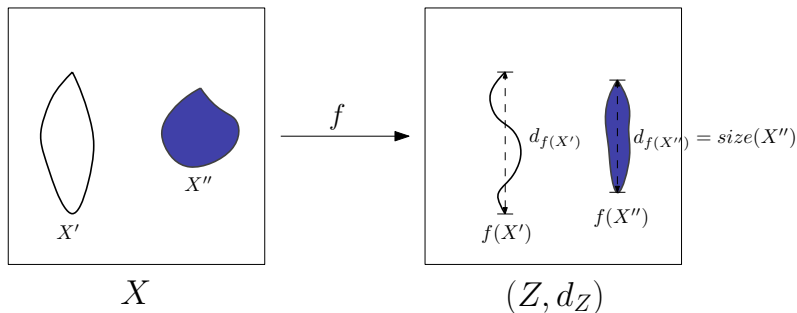# Persistent H_1-classes in multiscale mapper

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Persistent H_1-classes in MM

Theorem

Consider a H_1-persistence module of a multiscale mapper induced by a tower of path connected covers:

$$\mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_0})\big) \overset{s_{1*}}{\to} \mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_1})\big) \overset{s_{2*}}{\to} \cdots \overset{s_{n*}}{\to} \mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_n})\big)$$

Let $\hat{s}_{i*} = s_{i*} \circ s_{(i-1)*} \circ \cdots \circ \bar{\phi}_{\mathcal{U}_{\varepsilon_0}*}$. Then, $\hat{s}_{i*}$ renders the small classes of $H_1(X)$ trivial in $H_1(N(f^*\mathcal{U}_{\varepsilon_i}))$ as detailed in previous theorem.

Higher dimensional homology

- $\exists$ a metric d_δ on mapper $N(\mathcal{U})$ so that
  $d_{GH}((N(\mathcal{U}), d_\delta), (X, d_f)) \leq 5\delta$
  - convergence of Reeb space to mappers [MW16]
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- Persistence diagram of (X, d_f) and $MM(\mathfrak{U}, f)$ are close
- Persistence diagrams of mapper and multiscale mapper are similar under an appropriate map-induced metric

Conclusion/Questions

- Surjection on H_1-classes from space to nerve and between nerves

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Topological Analysis of Nerves

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- Surjection on H_1-classes from space to nerve and between nerves
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- Connection of MM to Reeb space? (answered)

Conclusion/Questions

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- What does persistence of MM compute? (answered)
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Conjecture: If t-wise intersections in $\mathcal{U}$ for all $t > 0$ have $\tilde{H}_{\leq k-t} = 0$, then is $\phi_{\mathcal{U}*}$ surjective for H_k?

Thank You