Homology through Interleaving

32 Concept of interleaving

A discrete set $P \subset \mathbb{R}^k$ is assumed to be a sample of a set $X \subset \mathbb{R}^k$, if it lies near to it which we can quantify with the Hausdorff distance $d_H(P, X)$. Observe that small $d_H(P, X)$ does not imply that $P$ necessarily lie in $X$. It can be around $X$.

Our goal is to examine Čech and Rips complexes built on top of $P$ for inferring the homology of $X$. We achieve this goal by the following steps:

1. Consider the distance function to $X$, $d_X : \mathbb{R}^k \to \mathbb{R}$, $x \mapsto d(x, X)$, and the distance function to the sample $P$, $d_P : \mathbb{R}^k \to \mathbb{R}$, $x \mapsto d(x, P)$.

2. Let $X_\alpha := d_X^{-1}(-\infty, \alpha]$ and $P_\alpha := d_P^{-1}(-\infty, \alpha]$ be the $\alpha$-offsets of $X$ and $P$ respectively. Observe that $P_\alpha$ is the union of a set of balls with centers in $P$ and radii $\alpha$.

3. Observe that, for sufficiently small $\alpha < \alpha'$, $X_\alpha$ and $X'_\alpha$ are homotopy equivalent. In fact, $\alpha'$ can be 0 when $X$ is a compact manifold with positive weak feature size.

4. Argue that the sequence of $X_\alpha$ and $P_\alpha$ interleave, that is, for appropriate $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_5$

$$X_{\alpha_1} \subseteq P_{\alpha_2} \subseteq X_{\alpha_3} \subseteq P_{\alpha_4} \subseteq X_{\alpha_5}. \quad (9)$$

5. Use Nerve theorem to establish that the Čech complex $\mathcal{C}^\alpha(P)$ which is the nerve of $P_\alpha$ is homotopy equivalent to $P_\alpha$. There exists an homotopy equivalence that commutes with the inclusions at the homology level. This will be clear later.

6. Because of 5, we have an interleaving sequence of homomorphisms at the homology level from the sequence in 9:

$$H(X_{\alpha_1}) \to H(\mathcal{C}^{\alpha_2}) \to H(X_{\alpha_3}) \to H(\mathcal{C}^{\alpha_4}) \to H(X_{\alpha_5}).$$

7. From the sequence in 6, one can derive that the image of $H(\mathcal{C}^{\alpha_2}) \to H(\mathcal{C}^{\alpha_4})$ is isomorphic to $H(X_{\alpha_3})$.

8. Now use the interleaving between Čech and Rips filtrations to derive that the persistent homology between two Rips complexes is isomorphic to the homology of an offset of $X$.

33 Data on a compact set

First we consider a point data $P$ that presumably samples a compact subset $X$ of $\mathbb{R}^k$. It is known that an offset $X_\alpha$ for any $\alpha > 0$ may not be homotopy equivalent to $X$ when $X$ is compact. So, in this case we will consider capturing the homology groups of an offset $X_\alpha$ of $X$. We will need the following definitions for stating the precise results.
Notes by Tamal K. Dey, OSU

**Definition 55.** Let $X \subset \mathbb{R}^k$ be a compact set. Let $M$ denote the medial axis of $X$ and $C$ be the set of critical points of the distance function $d_X : \mathbb{R}^k \to \mathbb{R}$, $x \mapsto d(x, X)$. The reach $\rho(X)$ and the weak feature size $\text{wfs}(X)$ are defined as:

$$\rho(X) = \inf_{x \in X} d(x, M)$$
$$\text{wfs}(X) = \inf_{x \in X} d(x, C).$$

The following result says that the offsets of $X$ remain homotopically equivalent and hence possess isomorphic homology groups as long as the intervals do not contain critical points of $d_X$.

**Proposition 50.** If $0 < \alpha < \alpha'$ are such that there is no critical value of $d_X$ in the closed interval $[\alpha, \alpha']$, then $X_\alpha$ deformation retracts onto $X_\alpha$. In particular, $H(X_\alpha) \cong H(X_{\alpha'})$.

We will need the following useful fact.

**Fact 11.** Given a sequence $A \to B \to C \to D \to E \to F$ of homomorphisms between finite-dimensional vector spaces, if $\text{rank}(A \to F) = \text{rank}(C \to D)$, then this quantity also equals the rank of $B \to E$. Similarly, if $A \to B \to C \to E \to F$ is a sequence of homomorphisms such that $\text{rank}(A \to F) = \text{dim} C$, then $\text{rank}(B \to E) = \text{dim} C$.

**Proposition 51.** Let $P$ be finite set in $\mathbb{R}^k$, such that $d_H(X, P) < \varepsilon$ for some $\varepsilon < \frac{1}{4}\text{wfs}(X)$. Then, for all $\alpha, \alpha' \in [\varepsilon, \text{wfs}(X) - \varepsilon]$ such that $\alpha' - \alpha \geq 2\varepsilon$, and for all $\beta \in (0, \text{wfs}(X))$, we have $H(X_\alpha) \cong \text{image } i_\alpha$, where $i_\alpha : H(P_\alpha) \to H(P_{\alpha'})$ is the homomorphism between homology groups induced by the canonical inclusion $i : P_\alpha \to P_{\alpha'}$.

**Proof.** Assume without loss of generality that $\varepsilon < \alpha < \alpha' - 2\varepsilon < \text{wfs}(X) - 3\varepsilon$, since otherwise we can replace $b$ by any $b' \in (d_H(X, P), b)$. From the hypothesis we deduce the following sequence of inclusions:

$$X_{\alpha - \varepsilon} \subseteq P_\alpha \subseteq X_{\alpha' + \varepsilon} \subseteq P_{\alpha'} \subseteq X_{\alpha' + \varepsilon}$$

By Proposition 50, for all $0 < \beta < \beta' < \text{wfs}(X)$, the canonical inclusion $X_\beta \to X_{\beta'}$ is a homotopy equivalence. As a consequence, Eq. (10) induces a sequence of homomorphisms between homology groups, such that all homomorphisms between homology groups of $X_{\alpha - \varepsilon}, X_{\alpha' + \varepsilon}, X_{\alpha' + \varepsilon}$ are isomorphisms. It follows then from Fact 11 that $i_\alpha : H(P_\alpha) \to H(P_{\alpha'})$ has same rank as these isomorphisms. Now, this rank is equal to the dimension of $H(X_\alpha)$, since the $X_\beta$ are homotopy equivalent to $X$ for all $0 < \beta < \text{wfs}(X)$. It follows that $\text{image } i_\alpha \cong H(X_\alpha)$, since our ring of coefficients is a field.

The above proposition relates the homology of $X_\alpha$ with the persistent homology between two union of balls. We can go to the nerve of the union of balls, that is, the Čech complexes if we know that the following diagram commutes. The downward vertical arrows are isomorphisms induced by the homotopy equivalence due to the nerve theorem. The horizontal arrows are induced by inclusions.

$$\begin{align*}
H(P_\alpha) & \xrightarrow{i_\alpha} H(P_{\alpha'}) \\
\downarrow h_\alpha & \quad \quad \downarrow h_{\alpha'} \\
H(C_\alpha(P)) & \xrightarrow{i_{\alpha'}} H(C_{\alpha'}'(P))
\end{align*}$$
Chazal and Oudot [1] showed that the above diagram commutes. Then, we have the following Proposition.

**Proposition 52.** Let $P$ be finite set in $\mathbb{R}^k$, such that $d_H(X, P) < \varepsilon$ for some $\varepsilon < \frac{1}{4}\text{wfs}(X)$. Then, for all $\alpha, \alpha' \in [\varepsilon, \text{wfs}(X) - \varepsilon]$ such that $\alpha' - \alpha \geq 2\varepsilon$, and for all $\lambda \in (0, \text{wfs}(X))$, we have $H(X_{\lambda}) \simeq \text{image } i_\ast$, where $i_\ast : \lambda(C^\alpha(P)) \to \lambda(C^{\alpha'}(P))$ is the homomorphism between homology groups induced by the canonical inclusion $i : C^\alpha(P) \to C^{\alpha'}(P)$.

**Theorem 53.** Let $P$ be a finite point set such that $d_H(X, P) < \varepsilon$ for some $\varepsilon < \frac{1}{9}\text{wfs}(X)$. Then, for all $\alpha \in [2\varepsilon, \frac{1}{3}(\text{wfs}(X) - \varepsilon)]$ and all $\lambda \in (0, \text{wfs}(X))$, we have $H(X_{\lambda}) \simeq \text{image } j_\ast$, where $j_\ast$ is the homomorphism between homology groups induced by the canonical inclusion $j : \mathcal{R}^\alpha(P) \to \mathcal{R}^{\lambda\alpha}(P)$.

**Proof.** We have already seen the following sequence:

$$ C^{\alpha/2}(P) \to \mathcal{R}^\alpha(P) \to C^{\alpha}(P) \to C^{2\alpha}(P) \to \mathcal{R}^{\lambda\alpha}(P) \to C^{4\alpha}(P). $$

(11)

Since $2\varepsilon \leq \alpha \leq \frac{1}{3}(\text{wfs} - \varepsilon)$, by Proposition 52 this sequence of inclusions induces a sequence of homomorphisms between homology groups, such that $H(C^{\alpha/2}(P)) \to H(C^{\lambda\alpha}(P))$ and $H(C^{\lambda\alpha}(P)) \to H(C^{2\alpha}(P))$ have ranks equal to $\dim H(X_{\lambda})$. Hence, by Proposition 11, rank $j_\ast$ is also equal to $\dim H(X_{\lambda})$. It follows that $\text{image } j_\ast \simeq H(X_{\lambda})$. \hfill $\square$

## 34 Data on manifold

When $X$ is a smooth manifold, the above results can be slightly improved. The main observation is that, for manifolds, the homology of union balls indeed become isomorphic to that of the manifold. Therefore, one does not need to go through the persistent homology between two Čech complexes to capture the homology of $X$. Instead, one can compute the homology of a single Čech complex to obtain that of $X$. The following result due to Niyogi, Smale, Weinberger [2] is key for this observation.

**Proposition 54.** Let $P \subset X$ be such that $d_H(X, P) \leq \varepsilon$ where $X \subset \mathbb{R}^k$ is a smooth manifold. If $2\varepsilon \leq \alpha \leq \sqrt[3]{\frac{5}{2}}\rho(X)$, there is a deformation retraction from $P_{\alpha}$ to $X$ which implies that $H(C^\alpha(P))$ is isomorphic to $H(X)$.

Now we can state a result similar to Theorem 53 where we use Proposition 54 instead of Proposition 52.

**Theorem 55.** Let $P$ be a finite point set such that $d_H(X, P) < \varepsilon$ for some $\varepsilon < \frac{1}{9}\text{wfs}(X)$. Then, for all $\alpha \in [2\varepsilon, \frac{1}{3}(\text{wfs}(X) - \varepsilon)]$ and all $\lambda \in (0, \text{wfs}(X))$, we have $H(X_{\lambda}) \simeq \text{image } j_\ast$, where $j_\ast$ is the homomorphism between homology groups induced by the canonical inclusion $j : \mathcal{R}^\alpha(P) \to \mathcal{R}^{\lambda\alpha}(P)$.

**Proof.** The proof is exactly same as the proof of Theorem 53 except that the sequence in (11) is shrunk by one Čech complex in the middle:

$$ C^{\alpha/2}(P) \to \mathcal{R}^\alpha(P) \to C^{\alpha}(P) \to \mathcal{R}^{2\alpha}(P) \to C^{2\alpha}(P). $$

Now, apply the second part of Proposition 11 to obtain the stated result. \hfill $\square$
References
