Locality Sensitive Hashing (LSH)

Huan Sun, CSE@The Ohio State University
11/14/2017
FINDING SIMILAR ITEMS

MMDS Secs. 3.2-3.4.
Slides adapted from: J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets,
http://www.mmds.org

FINDING SIMILAR ITEMS

Slides also adapted from Prof. Srinivasan Parthasarathy @OSU
Two Essential Steps for Similar Docs

1. **Shingling**: Convert documents to sets

2. **Min-Hashing**: Convert large sets to short signatures, while preserving similarity

Host of follow up applications
- e.g. Similarity Search
- Data Placement
- Clustering etc.

The set of strings of length $k$ that appear in the document.

**Signatures:** short integer vectors that represent the sets, and reflect their similarity.

Similarity Search
Data Placement
Clustering etc.
SHINGLING

Step 1: *Shingling*: Convert documents to sets

Document → Shingling

The set of strings of length $k$ that appear in the document
Define: Shingles

- A $k$-shingle (or $k$-gram) for a document is a sequence of $k$ tokens that appears in the doc
  - Tokens can be characters, words or something else, depending on the application
  - Assume tokens = characters for examples

Example: $k=2$; document $D_1 = \text{abcab}$
Set of 2-shingles: $S(D_1) = \{ab, bc, ca\}$
Similarity Metric for Shingles

- Document $D_1$ is a set of its $k$-shingles $C_1 = S(D_1)$

- Equivalently, each document is a 0/1 vector in the space of $k$-shingles
  - Each unique shingle is a dimension
  - Vectors are very sparse

- A natural similarity measure is the **Jaccard similarity**:
  \[ \text{sim}(D_1, D_2) = \frac{|C_1 \cap C_2|}{|C_1 \cup C_2|} \]
Motivation for Minhash/LSH

- Suppose we need to find similar documents among $N = 1$ million documents.

- Naïvely, we would have to compute pairwise Jaccard similarities for every pair of docs.

  - $N(N - 1)/2 \approx 5 \times 10^{11}$ comparisons
  - At $10^5$ secs/day and $10^6$ comparisons/sec, it would take 5 days

- For $N = 10$ million, it takes more than a year…
Step 2: **Minhashing**: Convert large variable length sets to short fixed-length signatures, while preserving similarity.
From Sets to Boolean Matrices

- **Rows** = elements (shingles)

- **Columns** = sets (documents)
  - 1 in row $e$ and column $s$ if and only if $e$ is a valid shingle of document represented by $s$
  - Column similarity is the Jaccard similarity of the corresponding sets (rows with value 1)
  - **Typical matrix is sparse!**

Note: Transposed Document Matrix

```
<table>
<thead>
<tr>
<th>Documents</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
```
Outline: Finding Similar Columns

- **So far:**
  - A document $\rightarrow$ a set of shingles
  - Represent a set as a boolean vector in a matrix

- **Next goal:** Find similar columns while computing small signatures
  - Similarity of columns $==$ similarity of signatures

Next Goal: Find similar columns, Small signatures

Naïve approach:

1) Signatures of columns: small summaries of columns
2) Examine pairs of signatures to find similar columns
   - Essential: Similarities of signatures and columns are related
3) Optional: Check that columns with similar signatures are really similar
Next Goal: Find similar columns, Small signatures

Naïve approach:

1) Signatures of columns: small summaries of columns
2) Examine pairs of signatures to find similar columns
   - Essential: Similarities of signatures and columns are related
3) Optional: Check that columns with similar signatures are really similar

Warnings:

Comparing all pairs may take too much time: Job for LSH
   - These methods can produce false negatives, and even false positives (if the optional check is not made)

Hashing Columns (Signatures) : LSH principle

- **Key idea:** “hash” each column \( C \) to a small signature \( h(C) \), such that:
  - (1) \( h(C) \) is small enough that the signature fits in RAM
  - (2) \( \text{sim}(C_1, C_2) \) is the same as the “similarity” of signatures \( h(C_1) \) and \( h(C_2) \)

- **Goal:** Find a hash function \( h(\cdot) \) such that:
  - If \( \text{sim}(C_1, C_2) \) is high, then with high prob. \( h(C_1) = h(C_2) \)
  - If \( \text{sim}(C_1, C_2) \) is low, then with high prob. \( h(C_1) \neq h(C_2) \)

- Hash docs into buckets. Expect that “most” pairs of near duplicate docs hash into the same bucket!
Min-Hashing

- **Goal:** Find a hash function $h(\cdot)$ such that:
  - if $\text{sim}(C_1, C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
  - if $\text{sim}(C_1, C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

- **Clearly, the hash function depends on the similarity metric:**
  - Not all similarity metrics have a suitable hash function

- **There is a suitable hash function for the Jaccard similarity:** It is called **Min-Hashing**
Min-Hashing

- Imagine the rows of the boolean matrix permuted under random permutation \( \pi \).

- Define a “hash” function \( h_\pi(C) = \) the index of the first (in the permuted order \( \pi \)) row in which column \( C \) has value 1:
  \[
  h_\pi(C) = \min_\pi \pi(C)
  \]

- Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column.
Min-Hashing

- Imagine the rows of the boolean matrix permuted under random permutation $\pi$.

- Define a “hash” function $h_{\pi}(C) =$ the index of the first (in the permuted order $\pi$) row in which column $C$ has value 1:

  $$h_{\pi}(C) = \min_{\pi} \pi(C)$$

- Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column.
## Min-Hashing Example

**Permutation** $\pi$  

**Input matrix (Shingles x Documents)**

**Signature matrix $M$**

2nd element of the permutation is the first to map to a 1

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Min-Hashing Example

Permutation $\pi$  Input matrix (Shingles x Documents)  Signature matrix $M$

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

$2^{nd}$ element of the permutation is the first to map to a 1

$4^{th}$ element of the permutation is the first to map to a 1
Min-Hashing Example

Permutation $\pi$  Input matrix (Shingles x Documents)

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

$\text{Signature matrix } M$

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

2nd element of the permutation is the first to map to a 1

4th element of the permutation is the first to map to a 1

Note: Another (equivalent) way is to store row indexes or raw shingles (e.g. mouse, lion):

<table>
<thead>
<tr>
<th>1</th>
<th>5</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>
Min-Hash Signatures

- Pick K=100 random permutations of the rows

- Think of $\text{sig}(C)$ as a column vector
  
  $\text{sig}(C)[i] =$ according to the $i$-th permutation, the index of the first row that has a 1 in column $C$

  \[
  \text{sig}(C)[i] = \min (\pi_i(C))
  \]

- Note: The sketch (signature) of document $C$ is small $\sim 100$ bytes!

- We achieved our goal! We “compressed” long bit vectors into short signatures
Key Fact

For two sets \( A, B \), and a min-hash function \( mh_i() \):

\[
Pr[mh_i(A) = mh_i(B)] = Sim(A, B) = \frac{|A \cap B|}{|A \cup B|}
\]

Unbiased estimator for Sim using \( K \) hashes (notation policy – this is a different \( K \) from size of shingle)

\[
\hat{Sim}(A, B) = \frac{1}{k} \sum_{i=1}^{k} I[mh_i(A) = mh_i(B)]
\]
Key Fact

For two sets $A$, $B$, and a min-hash function $m_{hi}()$:

$$Pr[ m_{hi}(A) = m_{hi}(B) ] = Sim(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

Unbiased estimator for $Sim$ using $K$ hashes (notation policy – this is a different $K$ from size of shingle)

$$\hat{Sim}(A, B) = \frac{1}{k} \sum_{i=1:k} I[ m_{hi}(A) = m_{hi}(B) ]$$

The similarity of two signatures is the fraction of the hash functions in which they agree.
Min-Hashing Example

<table>
<thead>
<tr>
<th>Permutation $\pi$</th>
<th>Input matrix (Shingles x Documents)</th>
<th>Signature matrix $M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Similarities:

<table>
<thead>
<tr>
<th>Col/Col</th>
<th>Sig/Sig</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3</td>
<td>2-4</td>
</tr>
<tr>
<td>1-2</td>
<td>3-4</td>
</tr>
<tr>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Min-Hashing Example

![Image of Min-Hashing Example with tables and matrices]

**Permutation $\pi$**

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

**Input matrix (Shingles x Documents)**

\[
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

**Signature matrix $M$**

\[
\begin{array}{cccc}
2 & 1 & 2 & 1 \\
2 & 1 & 4 & 1 \\
1 & 2 & 1 & 2 \\
\end{array}
\]

**Similarities:**

<table>
<thead>
<tr>
<th>Col/Col</th>
<th>Sig/Sig</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3</td>
<td>0.75</td>
</tr>
<tr>
<td>2-4</td>
<td>0.75</td>
</tr>
<tr>
<td>1-2</td>
<td>0</td>
</tr>
<tr>
<td>3-4</td>
<td>0</td>
</tr>
</tbody>
</table>

The Min-Hash Property

- Choose a random permutation $\pi$
- Claim: $\Pr[h_{\pi}(C_1) = h_{\pi}(C_2)] = \text{sim}(C_1, C_2)$
- Why?

One of the two cols had to have 1 at position $y$
The Min-Hash Property

- Choose a random permutation $\pi$
- **Claim:** $\Pr[h_{\pi}(C_1) = h_{\pi}(C_2)] = sim(C_1, C_2)$
- **Why?**
  - Let $X$ be a doc (set of shingles), $y \in X$ is a shingle

One of the two cols had to have 1 at position $y$
The Min-Hash Property

- Choose a random permutation $\pi$
- Claim: $\Pr[h_{\pi}(C_1) = h_{\pi}(C_2)] = \text{sim}(C_1, C_2)$
- Why?
  - Let $X$ be a doc (set of shingles), $y \in X$ is a shingle
  - Then: $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$
    - It is equally likely that any $y \in X$ is mapped to the min element

One of the two cols had to have 1 at position $y$
The Min-Hash Property

- Choose a random permutation $\pi$
- **Claim:** $\Pr[h_\pi(C_1) = h_\pi(C_2)] = \text{sim}(C_1, C_2)$
- **Why?**
  - Let $X$ be a doc (set of shingles), $y \in X$ is a shingle
  - Then: $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$
    - It is equally likely that any $y \in X$ is mapped to the min element
  - Let $y$ be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$
  - Then either: $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, or $\pi(y) = \min(\pi(C_2))$ if $y \in C_2$

One of the two cols had to have 1 at position $y$
The Min-Hash Property

- **Choose a random permutation** $\pi$
- **Claim:** $\Pr[h_\pi(C_1) = h_\pi(C_2)] = \text{sim}(C_1, C_2)$
- **Why?**
  - Let $X$ be a doc (set of shingles), $y \in X$ is a shingle
  - **Then:** $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$
    - It is equally likely that any $y \in X$ is mapped to the min element
  - Let $y$ be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$
  - **Then either:**
    - $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, or
    - $\pi(y) = \min(\pi(C_2))$ if $y \in C_2$

  - So the prob. that both are true is the prob. $y \in C_1 \cap C_2$
  - $\Pr[\min(\pi(C_1)) = \min(\pi(C_2))] = |C_1 \cap C_2| / |C_1 \cup C_2| = \text{sim}(C_1, C_2)$

One of the two cols had to have 1 at position $y$
The Min-Hash Property (Take 2: simpler proof)

- **Choose a random permutation** $\pi$
- **Claim:** $\Pr[h_\pi(C_1) = h_\pi(C_2)] = \text{sim}(C_1, C_2)$
- **Why?**
  - Given a set $X$, the probability that any one element is the min-hash under $\pi$ is $1/|X| \leftarrow (0)$
    - It is equally likely that any $y \in X$ is mapped to the min element
  - Given a set $X$, the probability that one of any $k$ elements is the min-hash under $\pi$ is $k/|X| \leftarrow (1)$
The Min-Hash Property (Take 2: simpler proof)

- **Choose a random permutation** $\pi$

- **Claim:** $\Pr[h_\pi(C_1) = h_\pi(C_2)] = \text{sim}(C_1, C_2)$

- **Why?**
  - Given a set $X$, the probability that any one element is the min-hash under $\pi$ is $1/|X|$ $\leftarrow (0)$
    - It is equally likely that any $y \in X$ is mapped to the *min* element
  - Given a set $X$, the probability that one of any $k$ elements is the min-hash under $\pi$ is $k/|X|$ $\leftarrow (1)$
  - For $C_1 \cup C_2$, the probability that any element is the min-hash under $\pi$ is $1/|C_1 \cup C_2|$ (from 0) $\leftarrow (2)$
  - For any $C_1$ and $C_2$, the probability of choosing the same min-hash under $\pi$ is $|C_1 \cap C_2|/|C_1 \cup C_2|$ $\leftarrow$ from (1) and (2)
Similarity for Signatures

- We know: $\Pr[h_\pi(C_1) = h_\pi(C_2)] = \text{sim}(C_1, C_2)$
- Now generalize to multiple hash functions

- The similarity of two signatures is the fraction of the hash functions in which they agree

- Note: Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures
Min-Hashing Example

Permutation \( \pi \)

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Input matrix (Shingles x Documents)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Signature matrix \( M \)

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Similarities:

<table>
<thead>
<tr>
<th>Similarities</th>
<th>1-3</th>
<th>2-4</th>
<th>1-2</th>
<th>3-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Col/Col</td>
<td>0.75</td>
<td>0.75</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sig/Sig</td>
<td>0.67</td>
<td>1.00</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Min-Hash Signatures

- Pick $K=100$ random permutations of the rows
- Think of $\text{sig}(C)$ as a $K \times 1$ column vector
- $\text{sig}(C)[i] = \min \ (\pi_i(C))$

Min-Hash Signatures

- Pick $K=100$ random permutations of the rows
- Think of $\text{sig}(C)$ as a $K \times 1$ column vector
- $\text{sig}(C)[i] =$ according to the $i$-th permutation, the index of the first row that has a 1 in column $C$

$$\text{sig}(C)[i] = \min (\pi_i(C))$$

- **Note:** The sketch (signature) of document $C$ is small $\sim 100$ bytes!

- **We achieved our goal!** We “compressed” long bit vectors into short signatures
Implementation Trick

- **Permuting rows even once is prohibitive**
- **Row hashing!**
  - Pick $K = 100$ hash functions $k_i$
  - Ordering under $k_i$ gives a random row permutation!
- **One-pass implementation**
  - For each column $C$ and hash-func. $k_i$, keep a “slot” for the min-hash value
  - Initialize all $\text{sig}(C)[i] = \infty$
  - Scan rows looking for 1s
    - Suppose row $j$ has 1 in column $C$
    - Then for each $k_i$:
      - If $k_i(j) < \text{sig}(C)[i]$, then $\text{sig}(C)[i] \leftarrow k_i(j)$

How to pick a random hash function $h(x)$?
**Universal hashing:**
$$h_{a,b}(x) = ((a \cdot x + b) \mod p) \mod N$$
where:
- $a,b$ ... random integers
- $p$ ... prime number ($p > N$)
Step 3: **Locality-Sensitive Hashing**: Focus on pairs of signatures likely to be from similar documents.
LSH: First Cut

- **Goal**: Find documents with Jaccard similarity at least \( s \) (for some similarity threshold, e.g., \( s = 0.8 \))

- **LSH – General idea**: Use a function \( f(x, y) \) that tells whether \( x \) and \( y \) is a candidate pair: a pair of elements whose similarity must be evaluated

- **For Min-Hash matrices**:
  - Hash columns of signature matrix \( M \) to many buckets
  - Each pair of documents that hashes into the same bucket is a candidate pair
Candidates from Min-Hash

- Pick a similarity threshold $s$ ($0 < s < 1$)

- Columns $x$ and $y$ of $M$ are a **candidate pair** if their signatures agree on at least fraction $s$ of their rows:

$$M(i, x) = M(i, y)$$

for at least frac. $s$ values of $i$

- We expect documents $x$ and $y$ to have the same (Jaccard) similarity as their signatures

LSH for Min-Hash

- **Big idea:** Hash columns of signature matrix $M$ several times

- Arrange that (only) similar columns are likely to hash to the same bucket, with high probability

- **Candidate pairs** are those that hash to the same bucket
Partition $M$ into $b$ Bands

Signature matrix $M$

$b$ bands

$r$ rows per band

One signature
Partition $M$ into Bands

- Divide matrix $M$ into $b$ bands of $r$ rows
- For each band, hash its portion of each column to a hash table with $k$ buckets
  - Make $k$ as large as possible
Partition $M$ into Bands

- Divide matrix $M$ into $b$ bands of $r$ rows

- For each band, hash its portion of each column to a hash table with $k$ buckets
  - Make $k$ as large as possible

- Candidate column pairs are those that hash to the same bucket for $\geq 1$ band

- Tune $b$ and $r$ to catch most similar pairs, but few non-similar pairs
Hashing Bands

Columns 2 and 6 are probably identical (candidate pair).

Columns 6 and 7 are surely different.
Simplifying Assumption

- There are **enough buckets** that columns are unlikely to hash to the same bucket unless they are identical in a particular band.

- Hereafter, we assume that “**same bucket**” means “**identical in that band**”.

- Assumption needed only to simplify analysis, not for correctness of algorithm.
Example of Bands

Assume the following case:

- Suppose 100,000 columns of $M$ (100k docs)
- Signatures of 100 integers (rows)
- Therefore, signatures take 40Mb
- Choose $b = 20$ bands of $r = 5$ integers/band

Goal: Find pairs of documents that are at least $s = 0.8$ similar
C₁, C₂ are 80% Similar

- **Find pairs of** \( \geq s=0.8 \) similarity, set \( b=20, \ r=5 \)

- **Assume:** \( \text{sim}(C₁, C₂) = 0.8 \)
  - Since \( \text{sim}(C₁, C₂) \geq s \), we want \( C₁, C₂ \) to be a candidate pair: We want them to hash to at least 1 common bucket (at least one band is identical)
C₁, C₂ are 80% Similar

- Find pairs of $\geq s=0.8$ similarity, set $b=20$, $r=5$

- **Assume**: $\text{sim}(C₁, C₂) = 0.8$
  - Since $\text{sim}(C₁, C₂) \geq s$, we want $C₁, C₂$ to be a candidate pair: We want them to hash to at least 1 common bucket (at least one band is identical)

- **Probability $C₁, C₂$ identical in one particular band**: $(0.8)^5 = 0.328$

- Probability $C₁, C₂$ are not similar in all of the 20 bands: $(1-0.328)^{20} = 0.00035$
  - i.e., about 1/3000th of the 80%-similar column pairs are false negatives (we miss them)

- We would find 99.965% pairs of truly similar documents
C₁, C₂ are 30% Similar

- Find pairs of $\geq s = 0.8$ similarity, set $b = 20$, $r = 5$

- Assume: $\text{sim}(C₁, C₂) = 0.3$

  - Since $\text{sim}(C₁, C₂) < s$, we want $C₁$, $C₂$ to hash to NO common buckets (all bands should be different)
Find pairs of $\geq s = 0.8$ similarity, set $b = 20$, $r = 5$

Assume: $\text{sim}(C_1, C_2) = 0.3$
- Since $\text{sim}(C_1, C_2) < s$ we want $C_1, C_2$ to hash to NO common buckets (all bands should be different)

Probability $C_1, C_2$ identical in one particular band: $(0.3)^5 = 0.00243$

Probability $C_1, C_2$ identical in at least 1 of 20 bands: $1 - (1 - 0.00243)^{20} = 0.0474$
- In other words, approximately 4.74% pairs of docs with similarity 30% end up becoming candidate pairs
- They are false positives since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold $s$
LSH Involves a Tradeoff

- **Pick:**
  - The number of Min-Hashes (rows of $M$)
  - The number of bands $b$, and
  - The number of rows $r$ per band

  to balance false positives/negatives

- **Example:** If we had only 15 bands of 5 rows, the number of false positives would go down, but the number of false negatives would go up.

Analysis of LSH – What We Want

Similarity $t = \text{sim}(C_1, C_2)$ of two sets

Probability of sharing a bucket

No chance if $t < s$

Similarity threshold $s$

Probability = 1 if $t > s$

What 1 Band of 1 Row Gives You

Remember:
Probability of equal hash-values = similarity

Similarity $t = \text{sim}(C_1, C_2)$ of two sets
$b$ bands, $r$ rows/band

- Columns $C_1$ and $C_2$ have similarity $t$
- Pick any band ($r$ rows)
  - Prob. that all rows in band equal = $t^r$
  - Prob. that some row in band unequal = $1 - t^r$
- Prob. that no band identical = $(1 - t^r)^b$
- Prob. that at least 1 band identical = $1 - (1 - t^r)^b$
What \( b \) Bands of \( r \) Rows Gives You

Probability of sharing a bucket

Similarity \( t = \text{sim}(C_1, C_2) \) of two sets

\[ t \sim (1/b)^{1/r} \]

At least one band identical

\[ 1 - (1 - t^r)^b \]
Example: $b = 20; r = 5$

- **Similarity threshold $s$**
- **Prob. that at least 1 band is identical:**

<table>
<thead>
<tr>
<th>$s$</th>
<th>$1-(1-s^r)^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>.006</td>
</tr>
<tr>
<td>.3</td>
<td>.047</td>
</tr>
<tr>
<td>.4</td>
<td>.186</td>
</tr>
<tr>
<td>.5</td>
<td>.470</td>
</tr>
<tr>
<td>.6</td>
<td>.802</td>
</tr>
<tr>
<td>.7</td>
<td>.975</td>
</tr>
<tr>
<td>.8</td>
<td>.9996</td>
</tr>
</tbody>
</table>
Picking $r$ and $b$: The S-curve

- Picking $r$ and $b$ to get the best S-curve
  - 50 hash-functions ($r=5$, $b=10$)

Blue area: False Negative rate
Green area: False Positive rate

LSH Summary

- Tune $M$, $b$, $r$ to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures.

- Check in main memory that candidate pairs really do have similar signatures.

- **Optional:** In another pass through data, check that the remaining candidate pairs really represent similar documents.
Summary: 3 Steps

- **Shingling**: Convert documents to sets
  - We used hashing to assign each shingle an ID

- **Min-Hashing**: Convert large sets to short signatures, while preserving similarity
  - We used similarity preserving hashing to generate signatures with property $\Pr[h_{\pi}(C_1) = h_{\pi}(C_2)] = sim(C_1, C_2)$
  - We used hashing to get around generating random permutations

- **Locality-Sensitive Hashing**: Focus on pairs of signatures likely to be from similar documents
  - We used hashing to find candidate pairs of similarity $\geq s$
Backup slides