Appendix

In this part, We prove Theorem 4.1, which says the $maxmin$ rate allocation problem P1 is NP-hard.

Proof: We reduce the Maximum Independent Set problem for unit disks in a plane (MIS-DISKS), which is known to be NP-hard [1], to P1. In MIS-DISKS, the objective is to select a maximum subset of non-overlapping disks.

Given an instance of the MIS-DISKS problem with M disks, we construct an instance of P1. Each disk corresponds to a femtocell with its femto-BS situated at the center of that disk. Each femtocell has two power levels, zero and unit power level, where the latter corresponds to a unit transmission range and unit interference range. An additional femtocell f is added that does not overlap with any other femtocell (See Figure 1). A macro-BS is added with a large enough coverage range that includes the covered regions of all the femtocells.

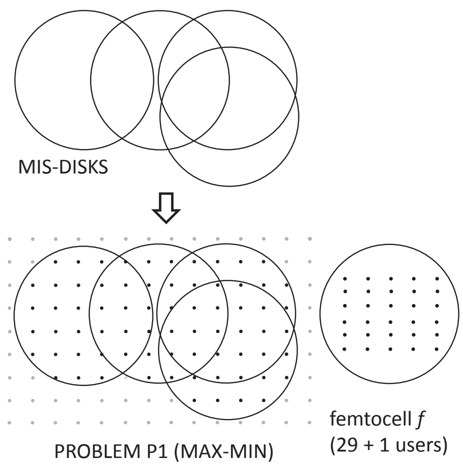

Fig. 1. Reduction for NP hardness. Here the radius of the circle is 3 times d, and $\eta(3)$ is known to be 29 [2]. So the additional femtocell f has 29+1 = 30 users. The dark dots represent the users. The gray dots are the lattice points outside the disks that were not selected to represent users.

Now consider a 2D lattice of points in the plane with a sufficiently high density (to be determined later). The lattice density will be chosen in such a way that the number of points within a unit disk is within a fixed range, say, $[K, K+\gamma]$, where $\gamma < \frac{K}{M}$. Each lattice point overlapping with any of the M femtocells will correspond to a user. In addition, $K+\gamma+1$ users are placed at any location within the range of femtocell f, thus making femtocell f the femtocell with the highest number of users.

If f is not in the optimum solution of the instance of P1, it can be added to increase the first term of expression Equation 4 with a lesser increase to the second term, leading to a resultant increase of the objective. So in the optimum solution to P1, f must be operating at unit power and the second term will have a value of $K+\gamma+1$.

Let S' be the set of disks corresponding to the femtocells other than f, that has a non-zero power allocation in the solution to P1. We claim that S' is a solution to the MIS problem. For the sake of contradition, let us assume that the optimum solution to the MIS problem, S, is such that $|S| > |S'|$. As the total number of users in range of the femtocells corresponding to S' is maximized, $K|S| \le (K+\gamma)|S'|$. Therefore, $\gamma \ge \frac{|S|-|S'|}{|S'|}K \ge \frac{|S|-|S'|}{M}K \ge \frac{K}{M}$. But in our construction $\gamma < \frac{K}{M}$, which is a contradiction. Thus, $|S| \le |S'|$, implying that S' is a solution to the MIS problem.

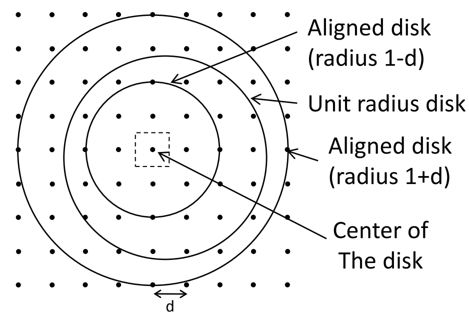

Fig. 2. Gauss's Circle Problem for Non-Lattice-disks: Alligned disks are lattice-disks. The square represents the region closest to the point at the center of the square.

Now we choose the appropriate value of lattice distance d ($d < 1$) such that the number of points within a unit disk is within the range $[K, K+\gamma]$. We say that a disk is a lattice-disk if its center coincides with a lattice point. If r is the ratio of the radius of the disk to the lattice distance, then using the Gauss' circle formula, the number of lattice points contained in it is represented as $\eta(r) = \pi r^2 + O(r)$ [2]. If the center of a unit disk is not aligned to a lattice point (Figure 2), then the number of lattice points will be within a range, $[K, K+\gamma]$. The nearest lattice point to any point on the plane is atmost at a distance of $\frac{d}{\sqrt{2}}$. So, centerd at that nearest lattice point, a lattice-disk of radius $1-d$ is fully contained within the unit disk, and a lattice-disk of radius $1+d$ will fully contain the unit disk. So the minimum number of lattice points for a unit disk, K will be atleast $\eta(\frac{1-d}{d})$, i.e., $K \ge \eta(\frac{1-d}{d}) = \pi(\frac{1}{d}-1)^2 + O(\frac{1}{d})$. Similarly, $K+\gamma$ will be atmost $\eta(\frac{1+d}{d})$, i.e., $K+\gamma \le \eta(\frac{1+d}{d}) = \pi(\frac{1}{d}+1)^2 + O(\frac{1}{d})$. Therefore, $\gamma \le \pi(\frac{1}{d}+1)^2 - \pi(\frac{1}{d}-1)^2 + O(\frac{1}{d}) = 4\pi(\frac{1}{d}) + O(\frac{1}{d})$. As $\frac{1}{d}$ increases, K grows quadratically but γ grows linearly. So for a sufficiently high value of $\frac{1}{d}$ (depends on M and the constants in $O(.)$), K will exceed γM, or, $\gamma < \frac{K}{M}$.

K and γ will both be polynomials in M. So, the total number of users created in this reduction is polynomial and the reduction is polynomial time, thus completing the proof. ■

References

[1] B.N. Clark, C.J. Colbourn, and D.S. Johnson. Unit Disk Graphs. *Discrete Mathematics*, 86:165–177, 1990.

[2] G.H. Hardy. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work.* AMS Chelsea Publishing, New York, 3 edition, 1999.