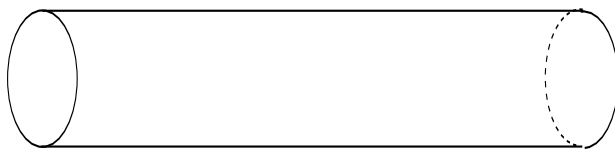


## Lecture 2 - First order linear PDEs and PDEs from physics

I mentioned in the first class some basic PDEs of first and second order. Today we illustrate how they come naturally as a model of some basic phenomena.

1.  $u_t + cu_x = 0$       Transport equation (simple transport)
2.  $u_{tt} - c^2 u_{xx} = 0$       Wave equation (vibrating string)
3.  $u_t = k\Delta u$       Parabolic equation (heat equation and diffusion)
4.  $\Delta u = 0$       Elliptic equation (stationary wave and diffusion)
5.  $-iu_t = \Delta u$       Schrödinger equation (Hydrogen atom)

- Derivation of (1):



Pipe of infinite length

Direction of water flow

Imagine putting a drop of ink in the pipe. Let  $u(x, t)$  be the concentration or density of ink at point  $x$  at time  $t$ .

But how do we describe it?

Fix an interval  $[0, b]$ .

$$\text{ink in } [0, b] \Big|_{\text{time}=t} = M = \int_0^b u(x, t) dx$$

Suppose the water moves at speed  $c$ , so at time  $h + t$  the same quantity of ink will be

$$M = \int_{ch}^{ch+b} u(x, t+h) dx$$

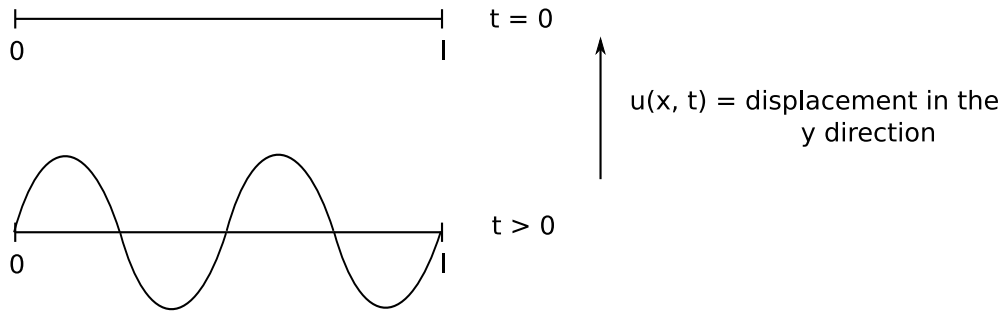
$$\int_0^b u(x, t) dx = \int_{ch}^{ch+b} u(x, t+h) dx$$

$$\text{Taking } \partial_b \Rightarrow u(b, t) = u(b+ch, t+h)$$

$$\text{Taking } \partial_h|_{h=0} \Rightarrow 0 = cu_x(b, t) + u_t(b, t)$$

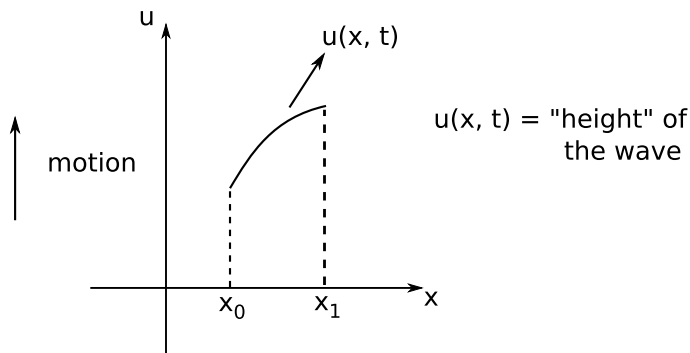
- Derivation of (2):

A typical example of wave motion in a plane is the motion of a string with fixed end points.



Why does it have this shape? We will see later.

Let's take a piece of it:



We will ignore all the forces on the string except for its tension. We will consider a perfect string in which for all  $x$  we have

$$\mathbf{T}(x, t) = T\mathbf{v}(x, t) \tag{1}$$

for constant  $T$  and  $\mathbf{v}(x, t)$  a unit tangent vector to the string at  $(x, t)$ .

$$\mathbf{v}(x, t) = \frac{(1, u_x)}{\sqrt{1 + u_x^2}}$$

We assume two facts:

1. The string has constant mass density  $\rho$ .
2. The tension does not change on the longitudinal ( $x$ ) component.

Now fix an interval of the string  $[x_0, x_1]$  and use Newton's law:

$$\begin{aligned} \mathbf{T}(x_1, t) - \mathbf{T}(x_0, t) &= m\mathbf{a}, T > 0 \\ T \cdot \frac{1}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1} &= 0 \\ T \cdot \frac{u_x}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1} &= \int_{x_0}^{x_1} \rho u_{tt}(x, t) dx \end{aligned} \tag{2}$$

Approximation:  $|u_x| \ll 1$ . Now consider the function  $f(z) = (1 + z^2)^{-1/2}$  and  $|z| \rightarrow 0$ . We can expand  $f(z)$  near  $z = 0$ :

$$\begin{aligned} f(z) &= (1 + z^2)^{-1/2} \\ f'(z) &= -\frac{1}{2}(1 + z^2)^{-3/2} \cdot 2z, f'(0) = 0 \\ f''(z) &= \frac{3}{2}z(1 + z^2)^{-5/2} - (1 + z^2)^{-3/2}, f''(0) = -1 \end{aligned}$$

So

$$\begin{aligned} f(z) &= f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \dots \\ f(z) &= 1 - \frac{1}{2}z^2 + \dots \end{aligned}$$

so for  $z$  small we can replace  $f(z)$  by 1 and then (2) becomes

$$\begin{aligned} T(x_1, t) - T(x_0, t) &= 0 \quad (T \text{ constant with respect to } x) \\ Tu_x \Big|_{x_0}^{x_1} &= \int_{x_0}^{x_1} \rho u_{tt}(x, t) dx \end{aligned} \quad (3)$$

Let's assume also that  $T = C$  is also constant with respect to time ( $t$ ) and differentiate (3) with respect to  $x$ :

$$\begin{aligned} (Tu_x)_x &= \rho u_{tt}(x, t) \\ u_{xx} &= c^2 u_{tt}, c^2 = \frac{\rho}{T} \end{aligned}$$

- Derivation of heat flow (3): Let  $D \subset \mathbb{R}^3$  be a region in space and let  $U(t)$  be the amount of energy in region  $D$ . (Heat refers to a change in energy of a region.) Let  $u(x, y, z, t)$  be the temperature. Then we can write:

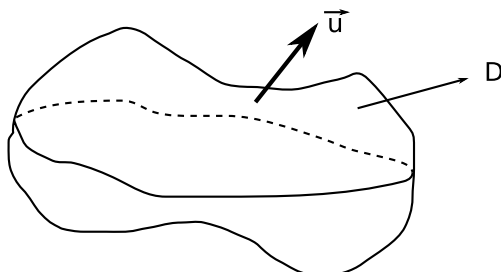
$$U(t) = \int_D c\rho u(x, y, z, t) dV$$

where  $\rho$  is the density of  $D$  (mass/volume) and  $c$  is the "specific heat", ( $dV \equiv dx dy dz$ ).

Clearly

$$\frac{d}{dt}U(t) = \int_D c\rho u_t dV.$$

(Here we are assuming that the material of  $D$  does not change in time.)



Assume, that outside the body the “space” is colder, and thus that the energy “flux” (direction of heat flow) will go from inside to outside and will “diffuse” through the boundary of  $D = \partial D$ . So also we have

$$\frac{d}{dt}U(t) = \int_{\partial D} \kappa(\mathbf{n} \cdot \nabla u) dS,$$

which is the heat transferred through the surface, and  $\kappa > 0$  is the thermal conductivity. By the Divergence Theorem

$$\int_{\partial D} \kappa(\mathbf{n} \cdot \nabla u) dS = \int_D \nabla \cdot (\kappa \nabla u) dV$$

so

$$c\rho u_t = \nabla \cdot (\kappa \nabla u)$$

If  $\kappa$  is constant, it follows that

$$u_t = \frac{\kappa}{c\rho} \Delta u, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

- Derivation of stationary wave or diffusion equation (4): Assume that in the heat and wave equations there is no change in time. The wave and heat equations reduce to

$$\Delta u = 0$$

Definition: Any function  $u$  such that  $\Delta u = 0$  is called harmonic.

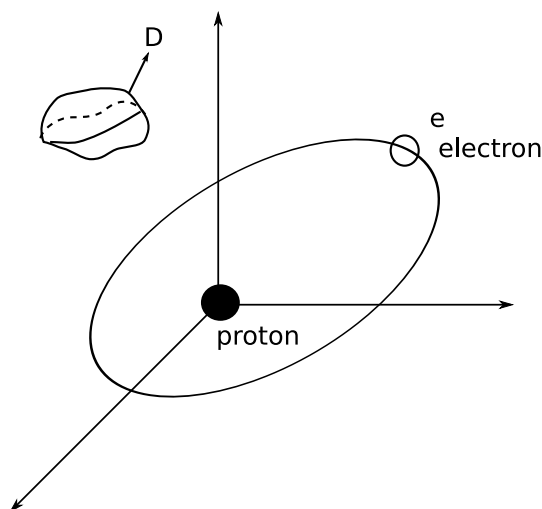
Remark: In one dimension all harmonic functions are linear:

$$f_{xx} = 0 \Rightarrow f_x = c_0 \Rightarrow f = c_0 x + c_1$$

- Schrödinger Equation for the Hydrogen Atom (5):

$$-i\hbar u_t = \frac{\hbar^2}{2m} \Delta u + \frac{e^2}{r} u$$

where  $\hbar$  is Planck’s constant over  $2\pi$ , and  $e^2/r$  is the potential energy of the system. Quantum mechanics tells us that  $u$  is a possible state (wavefunction) of the electron.



Let  $D$  be a region in  $\mathbb{R}^3$ . Then

$$\int_D |u|^2(x, y, z, t) dV \quad \text{is the probability that the electron is in } D \text{ at time } t,$$

$$\int_{\mathbb{R}^3} z |u|^2(x, y, z, t) dV \quad \text{is the expectation value of the electron } z \text{ coordinate at time } t,$$

if we normalize  $u$  such that

$$\int_{\mathbb{R}^3} |u|^2(x, y, z, t) dV = 1. \quad (4)$$

(We can perform similar calculations for  $x, y$ .)

The expected  $z$ -coordinate of the momentum is

$$\int_D dV -i\hbar \frac{\partial u}{\partial z}(x, y, z, t) \cdot \bar{u}(x, y, z, t)$$

Remark: It is possible to normalize  $u$  in the way described above, and furthermore the normalization of  $u$  does not change over time.

Proof:

$$\frac{d}{dt} \int_{\mathbb{R}^3} u \bar{u} dV = 2 \operatorname{Re} \int_{\mathbb{R}^3} u_t \bar{u} dV$$

$$u_t = i\hbar^{-1}(-i\hbar u_t) = \frac{i\hbar}{2m} \Delta u + \frac{i}{\hbar} \frac{e^2}{r} u$$

$$2 \operatorname{Re} \int_{\mathbb{R}^3} u_t \bar{u} dV = \underbrace{2 \operatorname{Im} \int_{\mathbb{R}^3} \frac{\hbar}{2m} \Delta u \cdot \bar{u} dV}_{(*)} + \underbrace{2 \operatorname{Im} \int_{\mathbb{R}^3} \hbar^{-1} \frac{e^2}{r} u \bar{u} dV}_{=0}.$$

Since  $(\Delta u) \bar{u} = \nabla u \cdot \overline{\nabla u}$ , and the latter is real,  $(*)$  is zero as well.

Exercise: Carefully derive the equation of motion for a string in a medium in which the

resistance is proportional to the velocity:

$$\begin{aligned} \text{Velocity } \mathbf{v}(x, t) &= \frac{d}{dt}(x, u(x, t)) = (0, u_t) \\ T(x, t) &= cu_t(x, t) \\ (cu_t u_x)_x &= \rho u_{tt} \end{aligned}$$

- Initial and Boundary Conditions

Even from the “resolution” of the transport equation we realized that in order to obtain only “one” solution to an equation we need to impose extra conditions. For example, at a certain time it should take a certain form or on a certain domain it should be of a certain form. The most common assumptions, which are justified by the original physical meaning of the PDE, are initial conditions and boundary conditions.

- Initial Conditions:

If we consider a transport equation it is reasonable to assign how the wave looks at time  $t = 0$ , so we write

$$u(0, x) = g(x).$$

We can do the same for the Schrödinger equation.

For the wave equation, just specifying  $u$  at  $t = 0$  is not enough, we also need the velocity at  $t = 0$  so we add

$$\dot{u}(0, x) = f(x)$$

- Boundary Conditions:

For the heat equation (stationary or not) it is important to specify how the solution looks like at  $\partial D$ , where  $D$  is the body. There are three well known boundary conditions:

- \*  $u|_{\partial D}$  - Dirichlet boundary conditions
- \*  $\frac{\partial u}{\partial n}|_{\partial D}$  - Neumann boundary conditions
- \*  $\frac{\partial u}{\partial n} + au$  - Robin condition

where  $a$  is a given function in  $x, y, z$  and  $\mathbf{n}$  is the unit normal vector to  $\partial D$ .

Definition: If the boundary condition is constantly equal to zero, then the condition is said to be homogeneous. If not, it is inhomogeneous.