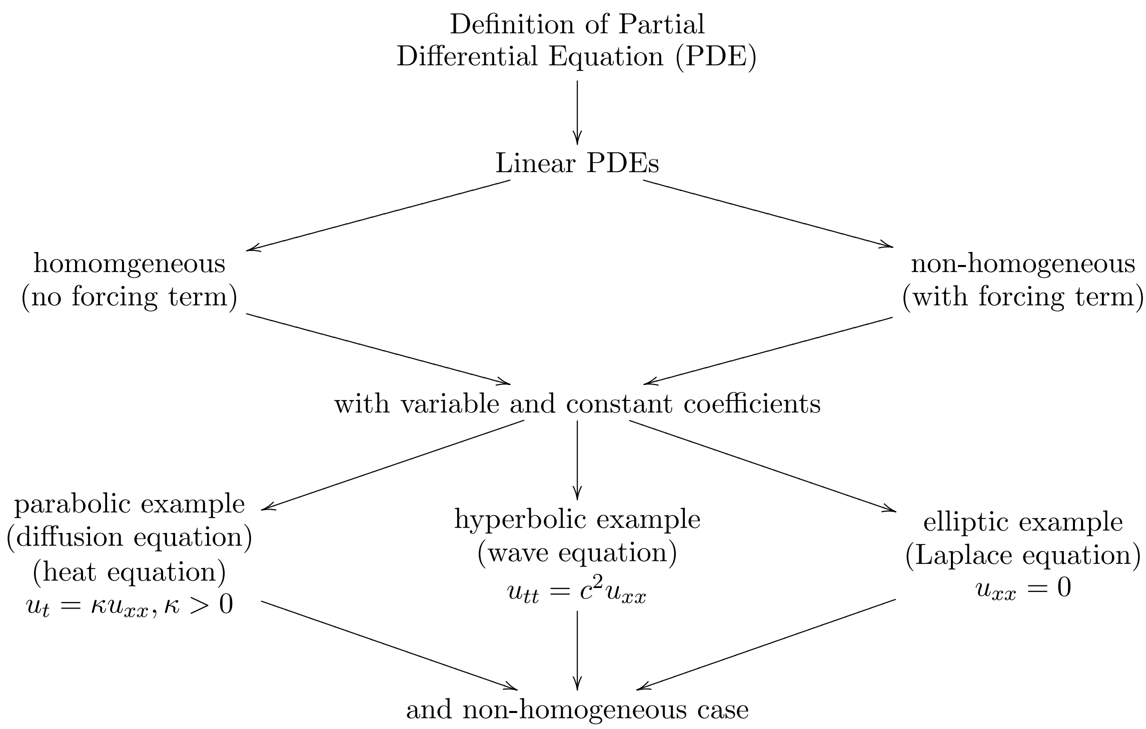
18.152 - Introduction to PDEs, Fall 2004 Prof. Gigliola Staffilani

Lecture 1 - Introduction and Basic Facts about PDEs

- The Content of the Course



In studying these examples of PDEs we will learn how to "impose conditions" to make the problem "well-posed", we will introduce fundamental mathematical concepts like "distributions", "Fourier Transform", and "Fourier Series". These tools are by now "classical", but still heavily used in the study of more complex PDEs, in particular, the nonlinear ones.

- What is a partial differential equation?

This is an equation involving a function $u(x_1, \ldots, x_n)$ of n variables and its partial derivatives up to order m:

$$F(u, u_{x_1}, \ldots, u_{x_n}, \ldots, u_{x_{i_1}x_{i_2}}, \ldots, u_{x_{i_1}x_{i_2}\cdots x_{i_m}}) = 0, i_j \in \{1, \ldots, n\}$$

This functional "defines" the equation by involving u and its partial derivatives.

In this case m is known as the order of the equation.

The variables $x_1, \ldots, x_n$ are independent and the variable $u(x_1, \ldots, x_n)$ is dependent. u is the unknown for the partial differential equation.

My notation for a partial derivative is

$$u_x = \partial_x u$$
$$u_{xy} = \partial_x \partial_y u = \partial_{xy} y$$

- Example:

$$u_{xy} + \cos(x)u_x + u_y^2 = 5 \tag{1}$$

is a partial differential equation of order 2 since u_{xy}, which is a double partial derivative, appears.

- If in the equation the unknown function u and its derivatives appear multiplied only by constants, like in

$$u_x + 4u_{xy} = u + x^2 \tag{2}$$

then the equation is said to be an "equation with constant coefficients".

If there are also other known functions involved as factors like in (1) (i.e. $\cos x$) the equation is said to be of variable coefficients.

- Linear and nonlinear equations

One can write a PDE as an operator $\mathcal{L}$ acting on u:

$$\mathcal{L}u = g(x_1, \dots, x_n) \tag{3}$$

($g(x_1, \dots, x_n)$ will be abbreviated $g(x)$.)

The operator $\mathcal{L}$ represents what one has to "do" in order to obtain the equation. For example, in (1):

$$\mathcal{L} = \underbrace{\partial_{xy}}_{\text{operator "take derivatives"}} + \overbrace{\cos x}^{\text{operator "multiplication by } \cos x\text{"}} + (\partial_y)^2$$

Definition: A PDE is linear if

$$\mathcal{L}(cu + bv) = c\mathcal{L}u + b\mathcal{L}v$$

for $b, c \in \mathbb{R}$. (This is linked to the definition of vector spaces.)

Definition: We say that a PDE of the form (3) is homogeneous if $g = 0$.

So $S = \{u | \mathcal{L}(u) = 0\}$ is a vector space.

Solutions to inhomogeneous linear equation are those of (3) with $g \neq 0$.

Suppose we find u_0 such that $\mathcal{L}(u_0) = g(x)$. Then $S = \{u + u_0 | \mathcal{L}(u) = 0\}$. In fact, if $v = u + u_0$ when $\mathcal{L}(u) = 0$, then

$$\begin{aligned} \mathcal{L}(v) = \mathcal{L}(u + u_0) &= \mathcal{L}(u) + \mathcal{L}(u_0) \\ &= 0 + g(x) \end{aligned}$$

so $\{u + u_0 | \mathcal{L}(u) = 0\} \subset S$, and on the other hand, if $w \in S$

$$\mathcal{L}(w - u_0) = \mathcal{L}(w) - \mathcal{L}(u_0) = g(x) - g(x) = 0$$

so $u = w - u_0 \in \{u | \mathcal{L}(u) = 0\} \Rightarrow S \subset \{u + u_0 | \mathcal{L}(u) = 0\}$

- Remark: If $\mathcal{L}(u) = 0$ and $\mathcal{L}$ is linear, then the linear combination of n solutions $\sum_{i=1}^{n} c_i u_i$ is still a solution:

$$\mathcal{L}(\sum_{i=1}^{n} c_i u_i) = \sum_{i=1}^{n} c_i \mathcal{L}(u_i) = 0 \tag{4}$$

(Clearly this is not true if the equation is linear but not homogeneous.)

(4) goes under the name of "superposition".

- Examples:
  - In (1), when $\mathcal{L} = \partial_{xy} + \cos x + (\partial_y)^2, \mathcal{L}$ is not linear because taking the square is not a linear procedure, that is, $(u+v)^2 \neq u^2 + v^2$ for $u, v \neq 0$.
  - In (2), where $\mathcal{L} = \partial_x + 4\partial_{xy} - 1, \mathcal{L}$ is a linear operator, so (2) is linear and non-homogeneous.

- Some well known equations

  1. $u_x + u_y = 0$ (transport)
  2. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
  3. $u_t + u_{xxx} + uu_x = 0$ (KdV equation)
  4. $u_t - iu_{xx} = 0$ (Scrhödinger equation)

- Remarks:

  1. is linear homogeneous 1st order
  2. is linear homogeneous 2nd order
  3. is nonlinear homogeneous 3rd order
  4. is linear homogeneous 2nd order

- How do we solve a PDE?

  There is no general rule that works to solve all PDEs, though clearly linear homogeneous PDEs are easier to solve than non-homogeneous or non-linear PDEs. One basic idea to keep in mind is how to solve ODEs:

  For example: $u_x - u = 0$ Find $u(x, y)$.

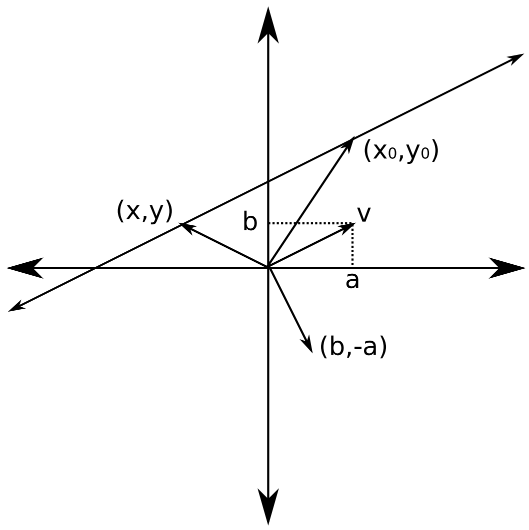
  By ignoring y, we get that $u(x, y) = C(y)e^x$ is a solution for any function $C(y)$.

- Solution of 1st order linear homogeneous equations in $\mathbb{R}^2$

  Consider the simple homogeneous equation with constant coefficients:

$$\begin{aligned} au_x + bu_y &= 0 \\ \Leftrightarrow (a,b)\cdot\nabla u &= 0 \end{aligned} \tag{5}$$

  The latter equation says that the directional derivative of u in the direction of $\mathbf{v} = (a,b)$ is zero. This means that the function $u(x,y)$ remains constant on lines in the direction of (a,b). The equations of such lines are



$$\begin{aligned} (x-x_0, y-y_0)\cdot(b,-a) &= 0 \\ \{bx - ay = c \,|\, c \in \mathbb{R}\} \end{aligned}$$

  and are called the characteristic lines for (5). If we think of a line as a function $y(x)$, then $\frac{d}{dx}y = \frac{b}{a}(a \neq 0)$.

  Now if u does not change along these lines, then

$$\begin{aligned} u(x,y)\Big|_{bx-ay=c} &= f(c) \\ \Rightarrow u(x,y) &= f(bx-ay) \end{aligned}$$

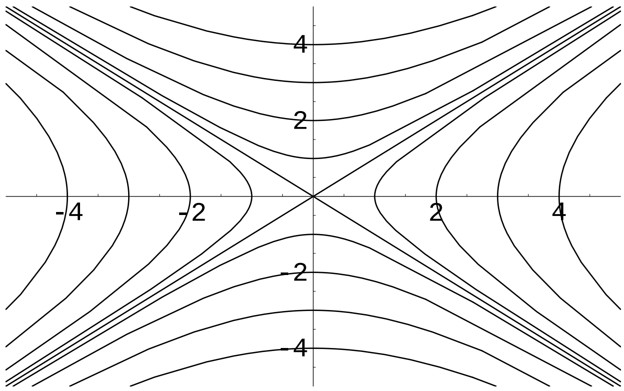
  If one wants a more precise description of f then some conditions must be specified.

- Now let's consider the case of variable coefficients. For example,

$$yu_x + xu_y = 0 \tag{6}$$

  As above, this means that u is constant along "curves" that have tangent vectors (y,x). So, similarly to the line, if the curve is represented by $y(x)$, then

$$\begin{aligned}
\frac{d}{dx}y &= \frac{x}{y} \\
\frac{dy}{dx}y &= x \\
\frac{d}{dx}\left(\frac{1}{2}y^2\right) &= x \\
\frac{1}{2}x^2 &= \frac{1}{2}y^2 + C
\end{aligned}$$



$y^2 - x^2 = c$ are the characteristic curves in this case. Thus $u(x,y) = f(y^2 - x^2)$. If we impose the condition $u(0,y) = e^{-y^2}$, then $f(y^2) = e^{-y^2} \Rightarrow f(t) = e^{-t}$, and thus $u(x,y) = e^{-(y^2-x^2)}$.

Finally,

$$a(x,y)u_x + b(x,y)u_y = 0 \tag{7}$$

can be solved as long as

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \tag{8}$$

can be solved as an ODE.

- Now we can solve $au_x + bu_y = c$ in full generality:

  First, find a special solution, use $u_0(x,y) = \alpha x$. Then

$$\begin{aligned}
a\alpha + b \cdot 0 = c \Rightarrow \alpha &= \frac{c}{a} \\
u_0(x,y) &= \frac{c}{a}x \\
u(x,y) &= f(bx - ay) + \frac{c}{a}x
\end{aligned}$$

- Consider the equation

$$au_x + bu_y + cu = 0 \tag{9}$$

  Find the set of solutions assuming $a \neq 0$: One solution is $u = 0$. Otherwise there will be at least some point at which $u \neq 0$. At such points we have:

$$\begin{aligned}
a\frac{u_x}{u} + b\frac{u_y}{u} + c = 0 \\
v \equiv \log u \Rightarrow v_x = \frac{u_x}{u}, v_y = \frac{u_y}{u} \\
\Rightarrow av_x + bv_y = c \\
\text{Thus from the previous: } v(x,y) &= f(bx - ay) - \frac{c}{a}x \\
\Rightarrow u(x,y) &= \exp\left(f(bx - ay) - \frac{c}{a}x\right) \\
&= e^{f(bx-ay)}/e^{\frac{c}{a}x}.
\end{aligned}$$