Lecture #17

We define the Fourier Series of a function $\phi(x)$ as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx \qquad n=0,1,\ldots$$

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx \qquad n=1,\ldots$$

Problem:

Solve
$$\begin{cases} u_{tt} = c^2 u_{xx} & 0<x<\pi \\ u_x(0,t) = u_x(\pi,t) = 0 & \\ u(x,0)=0 \quad u_t(x,0) = \cos^2 x & \end{cases}$$

Neumann problem

Solution

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{\cancel{\pi}} + B_n \sin\frac{n\pi ct}{\cancel{\pi}}\right) \cdot \cos\frac{n\pi x}{\cancel{\pi}}$$

and

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(*) $0 = \phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{\pi}$

(**) $\cos^2 x = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} B_n \cos nx$

(*) $\Rightarrow \quad A_n = 0 \quad n = 0, 1, \ldots$

Find FS for $\cos^2 x$ in cos

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 - \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} 1 - \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$$

$$\sin x \sin x = \frac{1}{2}\cos(x-x) - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$= \frac{1}{2\pi} \cdot 2\pi + \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos 2x \, dx \qquad 2x = \theta$$

$x =$

$$= 1 + \frac{1}{4\pi}\int_{-2\pi}^{2\pi} \cos\theta \, d\theta = 1$$

$$\cos^2 x = \frac{1}{2}(1 - \cos 2x) \qquad (***)$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \cos nx \, dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(1-\cos 2x)\cos nx \, dx =$$

$$= \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} \cos nx \, dx}_{=0} - \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} \cos 2x \cos nx \, dx}_{\substack{0 \text{ if } n\neq 2 \\ \frac{1}{2} \text{ if } n=2}}$$

$\cos^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ but this was found since the trigonometric identity

So ~~$B_n = 0$ $n \neq 2$ ... $\frac{1}{4c}$ $n \neq 2$~~ dx.

$$u(x,t) = \frac{1}{2}t + \frac{1}{4c}\sin nct \cos nx$$

Q.E.D.

<u>Remark 1</u>: In the previous lecture we defined

sin Fourier Series for ϕ on $[0,\ell]$

1) $$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}$$

Cos Fourier Series

2) $$\phi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos \frac{n\pi x}{\ell}$$ (3)

and a Full Fourier Series

3) $f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{\ell} + B_n \sin\frac{n\pi x}{\ell}\right)$

Question: when does one get 1), 2) or 3)?

Remark 2: $\sin\theta$ is an odd function since

$$\sin\theta = -\sin(-\theta)$$

the sum of odd functions is odd even when the sum is infinity so 1) can only occur when ϕ is odd

Similarly $\cos\theta$ is even, hence 2) can occur only when ψ is even

Finally 3) is needed when f is neither odd or even. In fact for x in $(-\ell, \ell)$

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$$
$$= \psi(x) + \phi(x)$$

$\psi(x)$ even part of f

$\phi(x)$ odd part of f and $\psi(x)$, $\phi(x)$

can be ~~de~~ decomposed respectively into cos Fourier series (5)
and sin Fourier series to get 3).

Remark 3: ~~xxxx~~ 3) can be defined either for a ~~function~~ "nice" function in $(-\ell,\ell)$ or for a function in $\mathbb{R}$ with period 2ℓ

~~xx~~ This remark should be used to understand the following boundary conditions

$u(0,t) = u(\ell,t)$ Dirichlet boundary corresponding to the odd functions

$u_x(0,t) = u_x(\ell,t)$ Neumann BC corresponding to the even functions

$u(\ell,t) = u(-\ell,t)$
$u_x(\ell,t) = u_x(-\ell,t)$ Periodic BC corresponding to periodic extension

Complex Form of Full F.S.

~~xxx~~ Consider f periodic of period 2ℓ, or f on $(-\ell,\ell)$

Consider the countable set of functions

$$\left\{ e^{\frac{i0\pi x}{\ell}},\ e^{\frac{i\pi x}{\ell}},\ e^{\frac{i2\pi x}{\ell}},\ \ldots \text{ for all } n \text{ integers} \right\}$$

~~xxxx~~ $e^{\frac{in\pi x}{\ell}}(-\ell) = e^{\frac{in\pi x}{\ell}}(\ell)$ periodic.

We want to write

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

and this comes from the fact that if we have two functions

$$f, g : \mathbb{R} \longrightarrow \mathbb{C} \quad \text{periodic of period } 2l$$

$$f \bullet g = \frac{1}{2l}\int_{-l}^{l} f(x)\, \overline{g}(x)\, dx$$

So

$$e^{in\pi x/l} \bullet e^{im\pi x/l} =$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{in\pi x/l - im\pi x/l}\, dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{i(n-m)\frac{\pi x}{l}}\, dx \quad \begin{cases} 1 & \text{if } n = m \\ 0 & \underbrace{\text{if } n \neq m}_{\text{periodicity!}} \end{cases}$$

Ex: Consider e^{x} on $(-l, l)$

a) Find the Complex Full FS.

$$C_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{x}\, e^{in\pi x/\ell}\, dx$$

$$= \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{x(1+in\pi/\ell)}\, dx$$

$$= \frac{1}{2\ell}\,\left.\frac{e^{x(1+in\pi/\ell)}}{1+\frac{in\pi}{\ell}}\right|_{-\ell}^{\ell}$$

$$= \frac{1}{2\ell}\,\left.\frac{e^{\frac{x}{\ell}(\ell+in\pi)}}{(1+in\pi)(1-in\pi)}(1-in\pi)\right|_{-\ell}^{\ell}$$

$$= \frac{(1-in\pi)}{2\ell(1+n^2\pi^2)}\left(e^{(\ell+in\pi)} - e^{-(-\ell+in\pi)}\right)$$

~~cos ℓ~~

~~$e^{\ell}(\cos n\pi + i\sin n\pi) - e^{-\ell}(\cos$~~

$$e^{\ell}(\cos n\pi + \underbrace{i\sin n\pi}_{=0}) - e^{-\ell}(\cos n\pi - \underbrace{i\sin n\pi}_{=0})$$

$$= 2\cos n\pi\left(\frac{e^{\ell}-e^{-\ell}}{2}\right) = 2\cos n\pi \sinh \ell$$

~~Because we started with a real function~~
~~we only care about the real part here~~

$$= \begin{cases} 2\sinh \ell & \text{if } n \text{ even} \\ -2\sinh \ell & \text{if } n \text{ odd} \end{cases}$$

$$C_n = \begin{cases} \dfrac{1}{2l(1+n^2\pi^2)}\, 2(-1)^n\, 2\sinh l \\ -\dfrac{i\,n\pi}{l(1+n^2\pi^2)}\,(-1)^n \sinh l \end{cases}$$

$$C_n = \alpha_n + i\beta_n$$

$$C_n e^{i\frac{n\pi x}{l}} = (\alpha_n + i\beta_n)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)$$

$$= \left(\alpha_n \cos\frac{n\pi x}{l} - \beta_n \sin\frac{n\pi x}{l}\right) + i\left(\beta_n \cos\frac{n\pi x}{l} + \alpha_n \sin\frac{n\pi x}{l}\right)$$

Because we started with a real function we get

$$e^x = \sum_{n=0}^{\infty}\left(\alpha_n \cos\frac{n\pi x}{l} - \beta_n \sin\frac{n\pi x}{l}\right)$$

$$\alpha_n = \frac{1}{l(1+n^2\pi^2)}(-1)^n \sinh l$$

$$-\beta_n = \frac{n\pi}{l(1+n^2\pi^2)}(-1)^n \sinh l$$

and this gives the full F.S.

<u>Orthogonality:</u>

Finite dimensional ~~vector space~~ Hilbert space

The standard example of finite dimension Hilbert space is

$$\mathbb{R}^n = \left\{ (a_1, \_, a_n) = \vec{v} \;/\; a_i \text{ in } \mathbb{R} \right\}$$

This is a vector space

$\vec{v_1}, \vec{v_2}$ in $\mathbb{R}^n$ for any a, b in $\mathbb{R}$

(*) $$a\vec{v_1} + b\vec{v_2} \text{ is still in } \mathbb{R}^n$$

We can define a dot product in it

$$\vec{v_1} \cdot \vec{v_2} = \langle v_1, v_2 \rangle = \sum_{i=1}^{n} a_i b_i$$

if $\vec{v_1} = (a_1, \_, a_n)$
$\vec{v_2} = (b_1, \_, b_n)$

The dot-product can be used to measure the size of a vector

(**) $$\|\vec{v_1}\| = \langle v_1, v_1 \rangle^{\frac{1}{2}}$$

We say that $\{\vec{v_1}, \_, \vec{v_n}\}$ is a orthonormal basis if

(***) $$\vec{v_i} \cdot \vec{v_j} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(*), (**), (***) is a Hilbert space

This gives

$$\vec{v}_i = \vec{e}_i = (0, 0, \ldots \underset{i}{1}, \ldots 0)$$

Given an orthonormal basis then any vector

$$\vec{v} = \sum_{i=1}^{n} a_i e_i$$

and $a_i = \langle \vec{v}, e_i \rangle$

Consider the space of functions in $(-\ell, \ell)$ s.t.

$$(*) \int_{-\ell}^{\ell} |f(x)|^2 \, dx < \infty$$

$$\left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \;\middle|\; (*) \text{ holds} \right\} = L^2(-\ell, \ell)$$

We can define the inner product

$$\langle f, g \rangle = \int_{-\ell}^{\ell} f(x) \, \overline{g}(x) \, dx$$

and

$$\| f \| = \langle f, f \rangle^{\frac{1}{2}}$$

We would like to find a countable set of

functions $\{ f_n \mid n \in \mathbb{Z} \}$

s.t. $\langle f_i, f_j \rangle = \delta_{ij}$ and

any $f(x) = \sum_{n=-\infty}^{\infty} A_n f_n \quad -l<x<l$.

Now we know what f_n are (up to some constants), but how were they found to start with?

~~the most difficult~~ In general one huge difficulty in finding an ∞ basis is the orthogonal condition

$$\langle f_i, f_j \rangle = 0 \quad i \neq j$$

Eigenfunctions and orthogonality

Consider the operator $A = -\frac{d^2}{dx^2}$ in (a, b)

and let $X_1(x)$ and $X_2(x)$ be two eigenfunctions

$$-X_i'' = \lambda_i X_i \qquad i=1,2 \qquad \lambda_1 \neq \lambda_2$$

~~with either Dirichlet or Neumann conditions = 0.~~

there is the following identity

$$-X_1'' X_2 + X_1 X_2'' = \left(-X_1' X_2 + X_1 X_2'\right)'$$

to check it just take derivative in RHS