Lecture # 13

Problem # 3 p. 64*

Recall the formula

$$v(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

Imposing the initial data on $x > c|t|$

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy$$

In our case $\quad v(x,t) = f(x+ct) \quad$ for $\quad \begin{array}{l} x>0 \\ t<0 \end{array}$

$$v_t(x,t) = c f'(x+ct)$$

hence $\quad t=0 \quad v(x,0) = f(x)$

$$v_t(x,0) = c f'(x)$$

So immediately

$$\frac{1}{2c}\int_{x-ct}^{x+ct} c f'(s)\,ds \underset{x > c|t|}{=} \frac{1}{2}\left[f(x+ct) - f(x-ct)\right]$$

and

$$\frac{1}{2c}\int_{ct-x}^{ct+x} c f'(s)\,ds = \frac{1}{2}\left[f(ct+x) - f(ct-x)\right]$$

$$x < c|t|$$

*Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York: Wiley, 3 March 1992. ISBN: 0471548685.

Now at $t>0$ if $x>ct$
then

$$v(x,t) = \frac{1}{2}\left[f(x+ct) + \cancel{f(x-ct)}\right] + \frac{1}{2}\left[f(x+ct) - \cancel{f(x-ct)}\right]$$

$$= f(x+ct)$$

if $0<x<ct$

$$v(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2}\left[f(ct+x) - f(ct-x)\right]$$

$$= f(x+ct) - f(ct-x)$$

To solve #4 first find a general formula for the Neumann problem and then proceed as above.

<u>Inhomogeneous</u> <u>diffusion</u> + <u>wave</u> <u>equations</u>

Consider the diffusion ~~equation~~ problem

$$\begin{cases} u_t - K u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \qquad -\infty<x<\infty \qquad t>0 \end{cases}$$

Physically this means that as time progresses there is a "source of heat or cooling" that changes the ~~[illegible]~~ the otherwise unadulterated dynamics of the temperature.

The solution of this problem is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y)\,dy$$

~~Expondition~~ where

$$S(x,t) = \frac{1}{\sqrt{4\pi K t}} e^{-\frac{x^2}{4Kt}}$$

How do we find this formula ?

The first analogy is with the ODE

$$(2)\begin{cases} y_t(t) + A y(t) = f(t) \\ y(t) = y_0 \text{ constant} \end{cases}$$

when A is a constant. Then thanks to the integrating factor e^{At} we have

$$y(t) = \underbrace{e^{-tA} y_0}_{\text{solution of linear problem}} + \overbrace{\int_0^t e^{(s-t)A} f(s)\,ds}^{\text{special solution!}}$$

On the other hand this is not just an analogy.

In fact if we take F.T. this becomes a proof

$$(3)\begin{cases} \hat{u}_t - k(+i\xi)^2 \hat{u} = \hat{f}(t,\xi) \\ \hat{u}(\xi,0) = \hat{\phi}(\xi) \end{cases}$$

For fixed ξ call $\hat{u}(\xi,t) = g(t)$, $\hat{\phi}(\xi) = y_0$

then ~~the~~ problem ~~(3)~~ (3) becomes exactly (2) with $A = +k\xi^2$

Then the solution is

$$g(t) = \hat{u}(\xi,t) = e^{-tk\xi^2}\hat{\phi}(\xi) + \int_0^t e^{(s-t)k\xi^2}\hat{f}(s,\xi)\,ds$$

so

$$u(x,t) = \mathcal{F}^{-1}\left(e^{-tk\xi^2}\hat{\phi}(\xi)\right) + \int_0^t \mathcal{F}^{-1}\left(e^{(s-t)k\xi^2}\hat{f}(s,\xi)\right)ds$$

Now recall that

$$\mathcal{F}^{-1}\left(e^{-tk\xi^2}\cdot\hat{\phi}(\xi)\right) = \mathcal{F}^{-1}\left(e^{-tk\xi^2}\right) * \mathcal{F}^{-1}(\hat{\phi})$$

$$= S(x,t) * \phi$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy +$$

$$\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(s,y)\,dy\,ds$$

Q.E.D.

Please check the proof in the book ~~that~~ ~~is~~

based on plugging in the guessed solution in the equation

Source on a half line

Consider

$$\begin{cases} u_t - k u_{xx} = f(t) & x > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = h(t) \end{cases}$$

This problem has two sources of heat (cold): $f(t)$ and $h(t)$.

Reduction to a known problem:

$$w(x,t) = u(x,t) - h(t)$$

Assume $h(t)$ is differentiable

$$w_t = u_t - h_t$$

$$w_t - k w_{xx} = u_t - h_t - k u_{xx} = f(t) - h_t(t)$$

New problem

$$\begin{cases} w_t - k w_{xx} = f(t) - h_t(t) \\ w(x,0) = \phi(x) - h(0) \\ w(0,t) = 0 \end{cases}$$

Now let ϕ_{ext} = odd extension of $\phi(x) - h(0)$
and f_{ext} = " " " $f(x,t) - h_t(t)$

One can now solve

$$\begin{cases} w_t - k w_{xx} = f(t) - h_t(t) \\ w(x,0) = \phi_{ext}(x) \end{cases}$$

$$w(x,t) = \int_{-\infty}^{\infty} S(x-y)\,\phi_{ext}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\,[f_{ext}]\,dy\,ds$$

and $u(x,t) = w\big|_{x>0}(x,t)$;

$$u(x,t) = -\int_{-\infty}^{0} S(x-y)\,[\phi(-y) - h(0)]\,dy$$

$$+ \int_0^{\infty} S(x-y)\,[\phi(y) - h(0))\,dy$$

$$- \int_0^t \int_{-\infty}^{0} S(x-y,t-s)\,[f(-y,s) - h_t(s)]\,ds$$

$$+ \int_0^t \int_0^{\infty} S(x-y,t-s)\,[f(y,s) - h_t(s)]\,ds$$

This solves problem # 1 in page 68*.

Wave with a Source

*Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York: Wiley, 3 March 1992. ISBN: 0471548685.

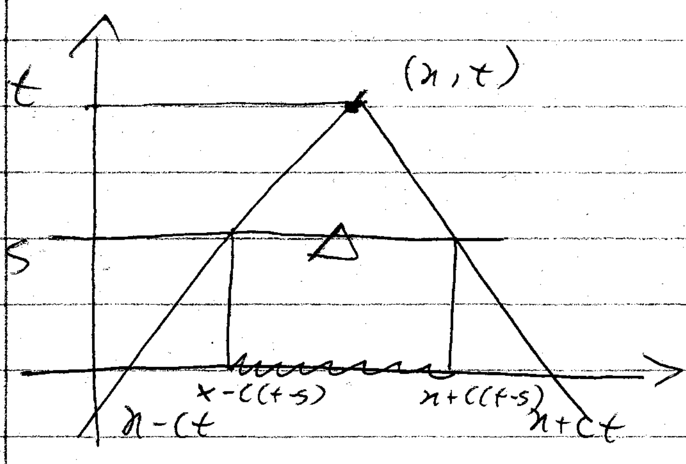
Consider the problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) & -\infty < x, t < \infty, \end{cases}$$

the solution for this problem is

(*)
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\iint_{\Delta} f\,dy\,ds$$

where $\Delta = [0,t] \times [x - c(t-s),\, x + c(t-s)]$



Remark: It makes sense that the "influence" of the source function $f(x,t)$ is only felt in the domain of influence.

Before we go to the arguments that bring us ~~the actual proof of~~ the formula for the solution let's use it to discuss well-posedness.

Well-posedness

Recall that well-posedness has three components

1) • Existence of solutions

2) • uniqueness of solutions

3) • stability of solutions

1) We check that $u(x,t)$ is sol.

$$u(x,t) = \underbrace{u_1(x,t)}_{\substack{\text{this solves eq.}\\ u_{tt}-c^2u_{xx}=0}} + \underbrace{u_2(x,t)}_{\substack{\text{this sol. eq.}\\ u_{tt}-c^2u_{xx}=0}} + u_3(x,t)$$

check that $u_3(x,t)$ also solves the eq.

$$\partial_{tt}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\,dy\,ds \quad , \quad \partial_{xx}(\quad) \text{ and continue}$$

Then by linearity u solves the eq.

For initial data
$$\begin{cases} t=0 & u(x,0)=\phi \\ t=0 & u_t(x,0)=\psi \quad \checkmark \end{cases}$$

2) Uniqueness: let u_1, u_2 be two solutions. then

Set $w = u_1 - u_2$ and

$$\begin{cases} w_{tt} - c^2 w_{xx} = f(t) - f(t) = 0 \\ w(x,0) = 0 \\ w_x(x,0) = 0 \end{cases}$$

$\Rightarrow$ energy argument $w = 0 \Leftrightarrow$
or causality principle $u_1 = u_2$

3) Stability: We define the "distance" of two functions

~~dist~~ $f_1(x,t)$ $f_2(x,t)$ $-\infty < x < \infty$, $0 \le t \le T$

$$\|f_1 - f_2\|_T = dist(f_1, f_2) = \max_{\substack{-\infty < x < \infty \\ 0 \le t \le T}} |f_1(x,t) - f_2(x,t)|$$

Also if there is no dependence on time

$$\|\phi_1 - \phi_2\| := \max_{-\infty < x < \infty} |\phi_1(x) - \phi_2(x)|$$

Assume for the initial data that

$$\|\psi_1 - \psi_2\|, \|\phi_1 - \phi_2\| < \delta, \quad \|f_1 - f_2\|_T < \delta$$

then $|u_1(x,t) - u_2(x,t)| =$

$$= \frac{1}{2}\Big| \frac{1}{2}\left[\phi_1(x+ct) - \phi_2(x+ct)\right] + \frac{1}{2}\left[\phi_1(x-ct) - \phi_2(x-ct)\right]$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} [\psi_1(s) - \psi_2(s)]\,ds + \frac{1}{2c}\int_\Delta [f_1(\cdot) - f_2(\cdot)]\,dy\,ds\Big|$$

$$\leq \frac{1}{2}\|\phi_1 - \phi_2\| + \frac{1}{2}\|\phi_1 - \phi_2\| + \frac{1}{2c}\|\psi_1 - \psi_2\|\, c\, 2|t|$$

$$+ \frac{1}{2c}\|f_1 - f_2\|_T \text{ Area}(\Delta)$$

$$\text{Area}(\Delta) = 2ct \cdot t = 2ct^2$$

$$\leq \delta + \delta T + \delta T^2 < \varepsilon$$

So for any $\varepsilon > 0$ we can find $\delta = \delta(T)$ s.t.

$$\|u_1 - u_2\|_T \leq \varepsilon.$$

<u>How do we find the formula</u> $(*)$?

Again we are going to use F.T. to the simpler problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(0,x) = 0 \\ u_t(0,x) = 0 \end{cases} \xrightarrow{F.T} \begin{cases} \hat{u}_{tt} - c^2 (i\xi)^2 \hat{u} = \hat{f}(\xi,t) \\ \hat{u}(0,\xi) = 0 \\ \hat{u}_t(0,\xi) = 0 \end{cases}$$

This is not much of a restriction because we know already the effect of the initial data & we have linearity. So if we fix ξ we get the ODE

$$\begin{cases} g_{tt} + c^2\xi^2 g = \hat{f}(\xi,t) \\ g(0)=0 \quad g_t(0)=0 \end{cases}$$

again the solution of this is easy:

$$\hat{u}(\xi,t) = g(t) = \int_0^t \frac{1}{|c\xi|} \sin (t-s)|\xi c| \, \hat{f}(\xi,s)\, ds$$

then one takes the inverse F.T.

$$u(x,t) = \int_0^t \mathcal{F}^{-1}\left(\frac{1}{|c\xi|} \sin(t-s)|\xi c|\right) * f(\cdot,s)\, ds$$

Using the fact that

$$\widehat{H(a-|x|)}\,(\xi) = \frac{2}{\xi} \sin a\xi$$

and some arithmetic manipulations also based on properties of sine one gets the formula

(*) !