$\boxed{\text{Lecture \# 7}}$

- Midterm next Th.
- Final exam Dec. 16, 9-12
- Next homework due next Tuesday.

Remember from last time:

u solve $\quad u_t = k u_{xx} \quad$ on $\quad R = [0,T] \times [0,\ell]$

we want to prove $\quad \max_R u = \max_{\partial_\sqcup R} u$

We introduced

$$v(x,t) = u(x,t) + \varepsilon x^2$$

we were left to prove that if

$$M = \max_{\partial_\sqcup R} u(x,t)$$

then $\quad (\ast\ast) \quad v(x,t) \leq M + \varepsilon \ell^2$ for all (x,t) in R

Proof of this fact

Observe that

$$v\Big|_{x=\ell} = u\Big|_{x=\ell} + \varepsilon \ell^2 \leq M + \varepsilon \ell^2$$

$$v\Big|_{x=0} = u\Big|_{x=0} \leq M$$

$$v\Big|_{t=0} = u\Big|_{t=0} + \varepsilon x^2 \leq M + \varepsilon \ell^2$$

so $\quad \max_{\partial_\sqcup R} v(x,t) \leq M + \varepsilon \ell^2$

Now we want to show that

$$\max_{\partial_\nu R} v = \max_{R} v$$

Assume (x_0, t_0) is a point s.t. $\max_R v = v(x_0, t_0)$

if (x_0, t_0) is in $\partial_\nu R \Rightarrow$ done

So then are two cases:

Case 1: (x_0, t_0) is in $\mathring{R}$ (interior of R)

Case 2: $t_0 = T$ and $0 < x_0 < l$

In the first case

$$u_x(x_0, t_0) = v_t(x_0, t_0) = 0$$

$$v_{xx}(x_0, t_0) \leq 0$$

$$v_t = u_t \qquad v_x = u_x + 2\varepsilon x \qquad v_{xx} = u_{xx} + 2\varepsilon$$

$$\underbrace{u_t}_{= v_t = 0} = k u_{xx} = k(\underbrace{v_{xx}}_{\leq 0} - 2\varepsilon) < 0$$

$\Rightarrow$ contradiction

In the second case:

$$f(x) = v(x, T) \qquad 0 < x < l$$

$$\lim_{\delta \to 0^+} \frac{v(x_0, T) - v(x_0 - \delta, T)}{\delta} = v_t(x_0, T) \geq c$$

and again for ⊛ we get a contradiction.

$\boxed{\text{Uniqueness}}$: Consider the boundary initial value problem

$$(*) \begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi \\ u(0,t) = g(t) \\ u(l,t) = h(t) \end{cases}$$

Uniqueness of the solution for $(*)$ means that for given ϕ, g, h there is only one solution for $(*)$, ϕ, g, h determines it completely.

Suppose there were 2 solutions for $(*)$, call them u_1 and u_2. Then $w = u_1 - u_2$ solves

$$(*)' \begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(l,t) = 0 \end{cases}$$

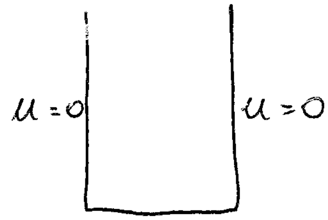
$w \equiv 0 \iff u_1 = u_2$

$\Uparrow$

By the max (min) principle $\quad \min_R w = \min_{\partial_L R} w = 0$

$\max_R w = \max_{\partial_L R} w = 0$

$\boxed{\text{Energy}}$ $E = \frac{1}{2}\int_0^l u(x,t)^2\,dx$ Assume $u(l,t) = u(0,t) = 0$



$$\frac{d}{dt}E = \frac{1}{\cancel{2}}\int_0^l u_t \cancel{2} u\,dx =$$

$$= \int_0^l u\,(+k\,u_{xx})\,dx$$

$$= +k\int_0^l u\,u_{xx}\,dx =$$

$$= +k\int_0^l \partial_x(u\,u_x)\,dx - k\int_0^l u_x^2\,dx$$

$$k\,u\,u_x\Big|_0^l = 0 \qquad -k\int_0^l u_x^2\,dx \le 0$$

so $E(t) \searrow$.

We can also prove uniqueness by using this fact:

$$0 \le E(t) \le E(0) = \frac{1}{2}\int_0^l w^2(x,0)\,dx = 0$$

so $E(t) = 0 \quad \forall\, t \in [0,T]$

$$\int_0^l [w(x,t)]^2\,dx = 0 \quad \Rightarrow w \equiv 0!$$

$\boxed{\text{Definition of Stability}}$: In general we say that a system is stable if "close" initial data generate "close" solutions.

To measure "closeness" we need a measure for distances of functions (5)
We now give an example of stability

Consider the diffusion equation $u_{t} = k u_{xx}$
~~…~~ We look at ~~…~~ two solutions ~~…~~ u_1 and u_2
$u_1(x, 0) = \phi_1$ and $u_2(x, 0) = \phi_2$.

~~…~~ suppose we define
$$dist(f, g) = \left(\int_0^l (f-g)^2(x)\, dx \right)^{\frac{1}{2}}.$$
~~Then by the energy inequality~~ Observe that

$w = u_1 - u_2$ solves the equation
and moreover $w\big|_{t=0} = \phi_1 - \phi_2$. Then by the energy inequality
$$\left(E(w)\right)^{\frac{1}{2}} = \left(\int_0^l (u_1 - u_2)^2(x)\, dx \right)^{\frac{1}{2}} \le \left(\int_0^l (\phi_1 - \phi_2)^2(x)\, dx \right)^{\frac{1}{2}}$$

So if ϕ_1 and ϕ_2 are "close" w.r.t. the distance above, then u_1 and u_2 are also "close" uniformly in time.

Now suppose we define the distance as
$$dist(f, g) = \max_{[0, l]} |f - g|$$

Assume that u_1 and u_2 solve the two problems

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = \phi_1 \\ u(0,t) = g \\ u(l,t) = f \end{cases} \qquad \begin{cases} v_t = k u_{xx} \\ v(x,0) = \phi_2 \\ v(0,t) = g \\ v(l,t) = f \end{cases}$$

Then $u_1 - u_2 = w$ solves
$$\begin{cases} w_t = k w_{xx} \\ w(x,0) = \phi_1 - \phi_2 \\ w(0,t) = w(l,t) \equiv 0 \end{cases}$$

Then by the maximum principle

$$\max_{R} w = \max_{\partial U} w = \max\left(\max_{[0,l]} \phi_1 - \phi_2 , 0 \right)$$

$$\max_{R} -w = \max_{\partial U} -w = \max\left(\max_{[0,l)} \phi_2 - \phi_1 , 0 \right)$$

so $\max_{[0,l]} |u_1 - u_2|(t) \le \max_{[0,l)} |\phi_1 - \phi_2|$

for all t.

Finding solutions for a diffusion eq

We here consider only the initial value problem

$$\begin{cases} u_t = k u_{xx} \\ u\big|_{t=0} = \phi(x) \end{cases}$$

The idea here is to find a source function $S(x,t)$ s.t. any solution for $(\ast)$ can be written in terms of S and the initial data like for the wave equation

For this process it is better to list few a-priori properties for the solution of a diffusion equation $(\square)$ $u_t = k u_{xx}$

a) <u>translation invariance</u>:

if $u(x,t)$ solves $(\square)$ then for any fixed y

$u(x-y,t)$ also solves $(\square)$

b) If u is a solution for $(\square)$ then all derivatives of any order of u also solves $(\square)$

c) A linear combinations of solutions for $(\square)$ is also a solution

d) If $w(x,t)$ is a solution then for any g "smooth" function

$$v(x,t) = \int w(x-y,t)\, g(y)\, dy$$

is also a solution for $(\square)$

e) Scaling: if $u(x,t)$ is a solution then $u(\lambda x, \lambda^2 t)$ " also " ".