Solutions

MATH 152, FALL 2004: MIDTERM #1

Problem #1 (15 pts)
Let $u_1(x,t)$ and $u_2(x,t)$ denote the solutions of the equation

$$u_t = u_{xx},$$

with initial and boundary conditions respectively $u_1(x,0) = g_1(x)$, $u_1(0,t) = f_1(t)$, $u_1(L,t) = h_1(t)$ and $u_2(x,0) = g_2(x)$, $u_2(0,t) = f_2(t)$, $u_2(L,t) = h_2(t)$. Assume that $g_1 \le g_2$, $f_1 \le f_2$ and $h_1 \le h_2$. Prove that $u_1 \le u_2$ in the set $R = [0,L] \times [0,\infty)$.
Hint: Let $w(x,t) = u_1(x,t) - u_2(x,t)$ and prove that $w(x,t) \le 0$ in R.

Problem #2 (20 pts)
This is an example of a heat problem with internal heat source for which the maximum principle does not hold. Consider

$$(1) \qquad \begin{cases} u_t = u_{xx} + 2(t+1) + x(1-x) \\ u(0,t) = 0, \ u(1,t) = 0 \\ u(x,0) = x(1-x) \end{cases}$$

for $0 < x < 1$ and $t > 0$.
a) Verify that $u(x,t) = (t+1)x(1-x)$ is a solution for (1). 7
b) Find the maximum M and the minimum m of the initial and boundary data. 7
c) Show that for all $t > 0$ the temperature distribution $u(x,t)$ exceeds M at a certain point inside the bar $[0,1]$. 6

Problem #3 (15 pts)
Consider the inhomogeneous problem

$$(2) \qquad \begin{cases} u_t = ku_{xx} + f(x,t) \\ u(0,t) = g(t), u(L,t) = h(t) \\ u(x,0) = \phi(x) \end{cases}$$

where $k > 0$, $0 < x < L$ and $t > 0$.
a) Are the boundary conditions of this problem of Dirichlet or Neumann type? 5
b) Prove the uniqueness of the solution for (2) using the energy method. 10

Problem #4 (20 pts)
Consider the initial value problem

$$(3) \qquad \begin{cases} 3u_{tt} + u_{xx} - 4u_{xt} = 0 \\ u(x,0) = x \\ u_t(x,0) = 0 \end{cases}$$

for $-\infty < x < \infty$ and $t > 0$.
a) Of what type (parabolic, hyperbolic, elliptic) is the equation in (3)? 6
b) Write the solution for (3). 14

Problem #5 (15 pts)

Use the coordinate method to solve

$$\begin{cases} u_x + u_y = u^2 \\ u(x,0) = h(x) \end{cases}$$

for $-\infty < x, y < \infty$.

Problem #6 (15 pts)

Consider the function

$$f_a(x) = \begin{cases} 1 + (2a)^{-1} & \text{for } |x| < a \\ 1 & \text{for } |x| > a. \end{cases}$$

a) Prove that f_a is a distribution if we define $(f_a, \phi) = \int f_a(x)\phi(x)\,dx$ for all test functions ϕ. 8

b) Show that in the sense of distributions

$$\lim_{a\to 0} f_a = 1 + \delta_0, \qquad 7$$

where δ_0 is the distribution such that $(\delta_0, \phi) = \phi(0)$ for all test functions ϕ.

Problem # 1

Define $w(x,t) = u_1(x,t) - u_2(x,t)$

The function w solves

$$\begin{cases} w_t = w_{xx} \\ w(x,0) = g_1(x) - g_2(x) \\ w(0,t) = f_1(t) - f_2(t) \\ w(L,t) = h_1(t) - h_2(t) \end{cases}$$

Clearly $w \leq 0$ on the boundary of R

Then by the max principle $w(x,t) \leq 0$ for all (x,t) in R

$\Rightarrow \quad w(x,t) = u_1(x,t) - u_2(x,t) \leq 0 \Rightarrow u_1(x,t) \leq u_2(x,t)$

for all (x,t) in R.

Problem # 2

a) $u_t(x,t) = x(1-x)$

$u_x(x,t) = (t+1)[1-x-x]$

$u_{xx}(x,t) = -2(t+1)$

$u_t(x,t) = x(1-x) = -2(t+1) + 2(t+1) + x(1-x)$ ✓

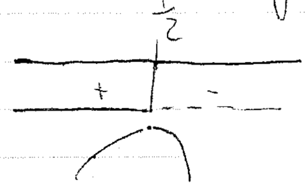
We now check the initial and boundary conditions

$u(x,t)\Big|_{x=0} = (t+1)x(1-x)\Big|_{x=0} = 0$ ✓

$$u(x,t)\Big|_{x=1} = (t+1)x(1-x)\Big|_{x=1} = 0 \quad \checkmark$$

$$u(x,t)\Big|_{t=0} = (t+1)x(1-x)\Big|_{t=0} = x(1-x) \quad \checkmark$$

b) let $f(x) = x(1-x)$ $\quad f'(x) = 1-2x \geq 0 \iff x \leq \frac{1}{2}$



so $\quad x = \frac{1}{2}$ max point $\quad f(\frac{1}{2}) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$ max

$\quad x=0,\ x=1$ min point $\quad f(0) = f(1) = 0$ min

Hence $\quad \min\limits_{\partial R} u = 0 = m \quad$ where $R = [0,1]\times[0,\infty)$

$$\max_{\partial R} u = \frac{1}{4} = M$$

c) $\ u(x,t) \leq u(\frac{1}{2},t) = (t+1)\frac{1}{4}$ for all x in $[0,1]$, $t>0$

and $\quad u(\frac{1}{2},t) > \frac{1}{4} = M$ for all $t>0$.

<u>Problem #3</u>

a) the boundary conditions are of Dirichlet type

b) let u_1 and u_2 be two solutions of the problem.

then $w = u_1 - u_2$ solves
$$\begin{cases} w_t = k\, w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = 0 \end{cases}$$

le define the energy

$$E(t) = \frac{1}{2}\int_0^L w^2(x,t)\,dx$$

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^L 2w(x,t)\,w_t(x,t)\,dx$$

$$= \int_0^L w(x,t)\,K\,w_{xx}(x,t)\,dx$$

$$= K\int_0^L \partial_x(w\,w_x)(x,t)\,dx$$

$$-k\int_0^L w_x^2(x,t)\,dx$$

$$= K\,\underbrace{w\,w_x\Big|_0^L}_{=0} - K\int_0^L w_x^2(x,t)\,dx$$

$$= -k\int_0^L w_x^2(x,t)\,dx \leq 0$$

So $E(t) \searrow \Rightarrow E(t) \leq E(0) = 0$

but also $0 \leq E(t)$ by definition

So $E(t) \equiv 0 \Rightarrow w^2 = 0 \Rightarrow w = 0$

$\Rightarrow u_1 = u_2$

Problem # 4

a) $a_{11} = 1 \quad a_{22} = 3 \quad a_{12} = -2$

$a_{12}^2 = 4 > a_{11}a_{22} = 3 \Rightarrow$ Hyperbolic

b) We can write the equation as

$$(\ast)\quad (\partial_x - 2\partial_t)^2 u - \partial_{tt}^2 u = 0$$

So if we introduce the new coordinates

$$\zeta = x$$
$$\xi = 2x + t$$

the equation $(\ast)$ can be written as

$$\partial_{\zeta\zeta}^2 u - \partial_{\xi\xi}^2 u = 0 \quad \Rightarrow$$

$$u(\zeta,\xi) = f(\zeta - \xi) + g(\zeta + \xi)$$

Changing back to the old variables

$$u(x,t) = f(x - 2x - t) + g(x + 2x + t)$$
$$= f(-x-t) + g(3x+t)$$

$$u(x,0) = f(-x) + g(3x) = x$$
$$u_t(x,t) = -f'(-x-t) + g'(3x+t)$$
$$u_t(x,0) = -f'(-x) + g'(3x) = 0$$

From the first identity we obtain differentiating:

$$(**)\quad \begin{cases} -f'(-x) + 3g'(3x) = 1 \\ -f'(-x) + g'(3x) = 0 \end{cases}$$

which combined with the last one gives

$$3g'(3x) - g'(3x) = 1$$

$$g'(3x) = \frac{1}{2} \quad \text{for all } x$$

If we change variables

$$g'(z) = \frac{1}{2} \quad \text{for all } z$$

$$g(z) = \frac{z}{2} + c_1$$

Similarly again from $(**)$

$$f'(-x) = g'(3x)$$

$$f'(-x) = \frac{1}{2} \quad \text{for all } x \text{ hence}$$

$$f(z) = \frac{1}{2} z + c_2$$

$$u(x,t) = \frac{1}{2}(-x-t) + c_2 + \frac{1}{2}(3x+t) + c_1$$

but $u(x,0) = -\frac{x}{2} + c_2 + \frac{3}{2}x + c_1 = x + c_1 + c_2$

$$\Rightarrow c_1 + c_2 = 0$$

So

$$u(x,t) = -\frac{x}{2} - \cancel{\frac{t}{2}} + \frac{3}{2}x + \cancel{\frac{t}{2}} = x$$

$$\boxed{u(x,t) = x}$$

Problem # 5

Introduce the new variables

$$x' = x + y$$

$$y' = x - y$$

Then the equation becomes

$$u_{x'} = \frac{u^2}{2}$$

$$\Leftrightarrow \quad \frac{u_{x'}}{u^2} = \frac{1}{2} \quad \Leftrightarrow \quad \left(-\frac{1}{u}\right)_{x'} = \frac{1}{2} \quad \text{Integrating}$$

$$-\frac{1}{u(x',y')} = \frac{1}{2}x' + c(y') \quad \text{for arbitrary } c(y')$$

$$\Leftrightarrow \quad u(x',y') = -\frac{2}{x' + 2c(y')}$$

changing back the variables

$$u(x,y) = -\frac{2}{x+y+2c(x-y)}$$

but

$$u(x,0) = -\frac{2}{x+2C(x)} = h(x)$$

whence $x + 2C(x) = -\frac{2}{h(x)}$

$$2C(x) = 2\left(-\frac{1}{h(x)} - \frac{x}{2}\right)$$

$$C(x) = -\left[\frac{1}{h(x)} + \frac{x}{2}\right] \quad \text{and}$$

$$u(x,y) = -\frac{2}{x+y-2\left[\frac{1}{h(x-y)} + \frac{x-y}{2}\right]}$$

$$= -\frac{2}{x+y-\frac{2}{h(x-y)}-(x-y)} = -\frac{1}{y-\frac{1}{h(x-y)}}$$

$$= \frac{h(x-y)}{1-y\,h(x-y)}.$$

Problem # 6

) Linearity : For any b, c in $\mathbb{R}$, φ, ψ in $\mathcal{D}$

$$(f_a, b\varphi + c\psi) = \int f_a(x) \left(b\varphi(x) + c\psi(x) \right) dx$$

$$= b \int f_a(x)\, \varphi(x)\, dx + c \int f_a(x)\, \psi(x)\, dx$$

↳ linearity of product and $\int$ $\quad = b(f_a, \varphi) + c(f_a, \psi)$

Continuity: let $\varphi_n \to \varphi$ in $\mathcal{D}$. This means that there exists an interval $[-M, M]$ s.t.

$$\varphi_n \big|_{[-M,M]^c} = \varphi \big|_{[-M,M]^c} \equiv 0$$

where $[-M, M]^c = (-\infty, -M) \cup (M, +\infty)$

and $\max\limits_{[-M,M]} |\varphi_n(x) - \varphi(x)| \xrightarrow[n \to \infty]{} 0$

then

$$(f_a, \varphi_n) - (f_a, \varphi) = \int f_a(x) \left(\varphi_n(x) - \varphi(x) \right) dx$$

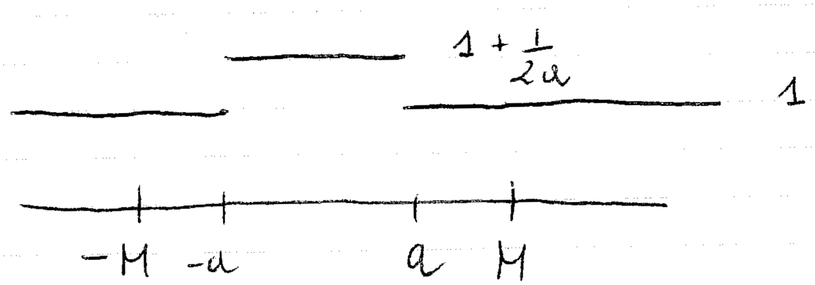
$$= \int_{-M}^{M} f_a(x) \left(\varphi_n(x) - \varphi(x) \right) dx$$

so

$$\left| (f_a, \varphi_n) - (f_a, \varphi) \right| \leq \int_{-M}^{M} |f_a(x)|\, dx \cdot \max_{[-M,M]} |\varphi_n - \varphi|$$

$\Rightarrow$ without loss of generality we can assume that $M > a$ so that

$$\int_{-M}^{M} |f_a(x)|\,dx = \int_{-M}^{M} 1\,dx + \int_{-a}^{a} \frac{1}{2a}\,dx = C_{M,a}$$



The point is that $C_{M,a}$ is independent of n!
So

$$|(f_a, \varphi_n) - (f_a, \varphi)| \le C_{M,a} \max_{[-M,M]} |\varphi_n - \varphi| \xrightarrow[n\to\infty]{} 0$$

$$\Longleftrightarrow \qquad (f_a, \varphi_n) \longrightarrow (f_a, \varphi)$$

b) Notice that $f_a(x) = 1 + \chi_a(x)$

where $\chi_a(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & |x| > a \end{cases}$

We already proved in class that $\chi_a \xrightarrow[a\to 0]{} \delta_0$
in the sense of distributions, hence $f_a \xrightarrow[a\to 0]{} 1 + \delta_0$!