$\boxed{\text{Lecture \# 22}}$

The Laplace equation

- If we consider stationary ~~wave eq~~ (independent of time) wave ~~eqn~~ or diffusion equations we will obtain

$$u_{xx} = 0 \quad \text{in } 1\text{-D}$$

$$\Delta u = 0 \quad \text{in } n\text{-D}$$

where if $u = u(x_1, \dots, x_n)$, then $\Delta u = \sum_{i=1}^{n} \partial_{x_i}^2 u$,
since both u_t and u_{tt} are zero. A

Definition: If $\Delta u = 0$ then u is called harmonic

- ~~The Lap~~ the inhomogeneous $\boxed{\text{Laplace equation}}$ takes the name of $\boxed{\text{Poisson equation}}$

$$\Delta u = f$$

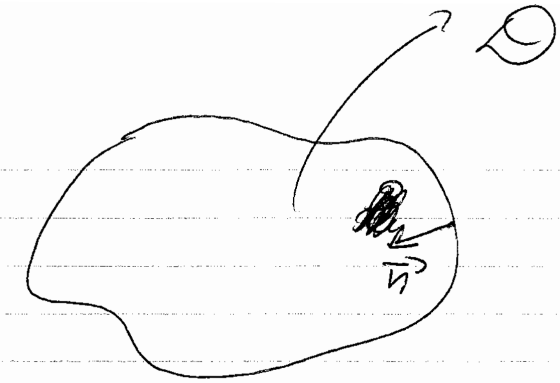
- The Boundary Value Problem that one solves is:
Given a domain $\mathcal{D}$ find u s.t.

Neumann　　　Robin

$$\begin{cases} \Delta u = f & \text{in } \mathcal{D} \\ u = h \text{ or } \dfrac{\partial}{\partial n} u = h \text{ or } \dfrac{\partial u}{\partial n} + a u = h & \text{in } \partial\mathcal{D} \end{cases}$$

Dirichlet

where $\frac{\partial}{\partial n}$ is the normal derivative to the boundary $\partial\mathcal{D}$ of $\mathcal{D}$.



$$\frac{\partial}{\partial \vec{n}} u = \langle \nabla u, \vec{n} \rangle$$

- The 1-D case is completely trivial since

$$u_{xx} = 0$$

gives the solution $u(x) = Ax + B$

and A and B are determined by the boundary conditions.

Also

$u_{xx} = f$ is solved by integrating twice.

So we will consider the problem of heat in 2-D.

Maximum Principle:

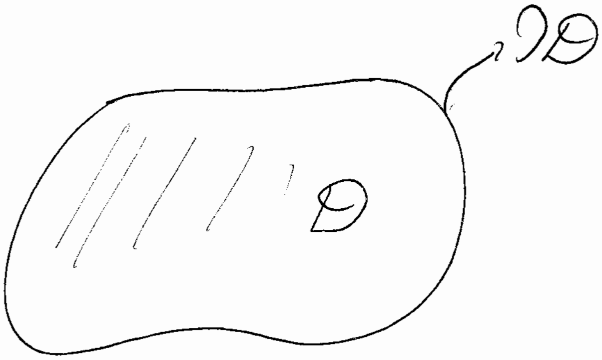
Let D be a connected bounded open set. Let u be an harmonic function in D and continuous all the way to ∂D. Then

a) $\max_{\bar{D}} u = \max_{\partial D} u \quad \left(\min_{\bar{D}} u = \min_{\partial D} u \right)$

If D is also connected and open, then

b) there are no max or min points in D unless $u \equiv$ const.

Proof: Assume for simplicity we are in $\mathbb{R}^2$



$$D \cup \partial D = \bar{D}$$

Remember the proof for diffusion equation:

If there is a point P in D s.t. $u(P) = \max_{\bar{D}} u$

then $\nabla u = 0$ and $u_{xx} \le 0$, $u_{yy} \le 0$

$$\Delta u = u_{xx} + u_{yy} \le 0$$

If we had < 0 then we would get the contradiction. But not there yet. So like for diffusion eq. let's modify a bit our function u.

Pick $\varepsilon > 0$ small and define

$$v(x,y) = u(x,y) + \varepsilon |(x,y)|^2$$

then v is continuous in $\bar{D}$, $\bar{D}$ is closed and bounded $\Rightarrow$ there exists $\max_{\bar{D}} v$ and $\min_{\bar{D}} v$

let's $P = \max_{\bar{D}} v$. Suppose P in D then

$$\nabla v = 0 \qquad v_{xx}, v_{yy} \le 0 \qquad (\ast)$$

$$v_{xx} = u_{xx} + \varepsilon (x^2 + y^2)_{xx} = u_{xx} + \varepsilon$$

$$v_{yy} = u_{yy} + \varepsilon$$

so $\quad v_{xx} + v_{yy} = u_{xx} + u_{yy} + 2\varepsilon = 2\varepsilon > 0 \quad$ (**)

Clearly (*) and (**) give a contradiction.

So P has to be on ∂D.

Now in general for any (x,y) in $\bar{D}$

$$u(x,y) \le v(x,y) \le v(P) = u(P) + \varepsilon |P|^2 < u(P) + \varepsilon \ell^2$$

where ℓ s.t. $B(0,\ell) \supset \bar{D}$ $\quad \le \max_{\partial D} u + \varepsilon \ell^2$

by letting $\varepsilon \to 0$

$$\max_{\bar{D}} u \le \max_{\partial D} u$$

$\ge$ trivial.

This prove the first part of Max. principle. Part b) will be proved next time.

Remark: For part a) the assumption "D connected" is not necessary.

Uniqueness of Dirichlet Problem on bounded domain

Assume that the problem $\begin{cases} \Delta u = f & D \\ u|_{\partial D} = h \end{cases}$ has two solutions u and v, then $w = u - v$ solves the problem

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w\big|_{\partial D} = 0 \end{cases}$$

If D is bounded then

$$\max_{\overline{D}} w = \max_{\partial D} w \equiv 0 \implies \max w \equiv 0$$

similarly $$\min_{\overline{D}} w = \min_{\partial D} w \equiv 0 \implies \min w = 0$$

$$0 \le w \le 0 \iff w \equiv 0 \iff v = u\,!$$

Here is a different way of proving uniqueness: Energy method

Consider the Dirichlet problem

$$\begin{cases} \Delta u = f \\ u\big|_{\partial D} = h \end{cases}$$

∂D smooth enough for divergence theorem

$w = u - v$, u, v two sol.

$$\cancel{\int_D w\,\Delta w = 0 = \int_D w\,\partial_{xx} w + w\,\partial_{yy} w = 0}$$

$$\cancel{= \int w\,\partial_x w + w\,\partial_y w\Big|_{\partial D} - \int_D (\partial_x w)^2 + (\partial_y w)^2}$$ as above

$$\begin{cases} \Delta w = 0 \\ w\big|_{\partial D} = 0 \end{cases}$$

Clearly $w \Delta w = 0$. Consider the vector

$$w \nabla w = w(\partial_x w, \partial_y w)$$
$$= (w \partial_x w, w \partial_y w)$$

By divergence theorem

$$\int_D \operatorname{div} \cdot \vec{f}\, \cancel{dx} = \int_{\partial D} \vec{f} \cdot \vec{n}\, ds$$

$$\operatorname{div}(w \nabla w) = \nabla \cdot (w \nabla w) = \nabla w \cdot \nabla w + w \Delta w$$

So

$$\int_D |\nabla w|^2 dx + \underbrace{\int_D w \Delta w\, dx}_{=0} = \int_{\partial D} w \cdot \nabla w \cdot \vec{n}\, ds$$

$$= \underbrace{\int_{\partial D} w \frac{\partial}{\partial \vec{n}} w\, ds}_{=0}$$

So $|\nabla w| \equiv 0 \Rightarrow \nabla w = 0 \Rightarrow w = \text{const}$

but $w\big|_{\partial D} = 0 \Rightarrow w \equiv 0$!

<u>Invariance of Laplacian</u>:

Assume we have a change of variables in $\mathbb{R}^2$

$$(x, y) \xrightarrow{A} (x', y')$$

We say that Δ is "invariant" under the transformation A is

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'} .$$

Theorem: Δ is invariant under rotations and translations

Proof:

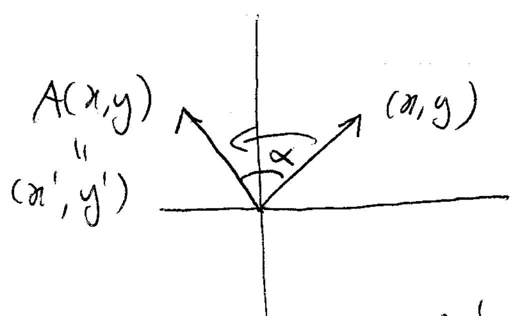
a) Invariance under translations: $(x, y) \longrightarrow (x+y) + (a, b)$

$$A(x,y) = (x+a,\ y+b) \implies \begin{array}{l} x' = x + a \\ y' = y + b \end{array}$$

$$\partial_{x'x'} = \partial_{xx}$$

$$\partial_{y'y'} = \partial_{yy}$$

b) Invariance under rotation:



rotation by angle α

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha$$

$$u_x = u_{x'}\cos\alpha + u_{y'}\sin\alpha \qquad u_{xx} = u_{x'x'}\cos^2\alpha + u_{x'y'}\cos\alpha\sin\alpha$$

$$u_y = -u_{x'}\sin\alpha + u_{y'}\cos\alpha \qquad + u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\sin^2\alpha$$

$$u_{y^*y^*} = + u_{x'x'} \sin\alpha^2 - u_{x'y'} \sin\alpha\cos\alpha$$
$$+ u_{y'x'} \sin\alpha\cos\alpha + u_{y'y'} \cos\alpha\cos\alpha$$

$$u_{x^*x^*} + u_{y^*y^*} = u_{x'x'}(\cos^2\alpha + \sin^2\alpha) + 2u_{x'y'}\cos\alpha\sin\alpha - 2u_{x'y'}\sin\alpha\cos\alpha$$
$$+ u_{y'y'}(\cos^2\alpha + \sin^2\alpha) = u_{x'x'} + u_{y'y'}$$

<u>Remark:</u> Invariance by rotation should indicate that in polar coordinate Δ is simple

<u>Laplace in polar coordinates</u>

$$x = r\cos\theta$$

$$y = r\sin\theta$$

~~the linear transform is~~

~~From the chain rule~~ the Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\cos\theta & r\cos\theta \end{pmatrix}$$

the inverse is

$$J^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}$$

Now if $u = u(r(x,y), \theta(x,y))$

$$\frac{\partial}{\partial x} u = \frac{\partial}{\partial r} u \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial y} u = \frac{\partial}{\partial r} u \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix} = J^{-1} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix}$$

So we obtain
$$\begin{cases} \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \end{cases}$$

$$\frac{\partial^2}{\partial x^2} = \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}\right]^2 = \cos^2 \frac{\partial^2}{\partial r^2} + \frac{\sin^2}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$- \cos\theta \frac{\partial}{\partial r}\left(\frac{\sin\theta}{r} \frac{\partial}{\partial \theta}\right) - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial}{\partial r}\right]$$

$$= \cos^2 \frac{\partial^2}{\partial r^2} + \frac{\sin^2}{r^2} \frac{\partial^2}{\partial \theta^2} + \cancel{\frac{\cos\theta}{r^2} \sin\theta} \frac{\partial}{\partial \theta}$$

$$+ \frac{\sin^2\theta}{r} \frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial y^2} = \left[\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \right]^2$$

$$= \sin^2 \frac{\partial^2}{\partial r^2} + \frac{\cos^2}{r^2} \frac{\partial^2}{\partial \theta^2} + \cancel{\frac{\sin\theta}{r^2} \cos\theta \frac{\partial}{\partial \theta}}$$

$$+ \frac{\cos\theta^2}{r} \frac{\partial}{\partial r}$$

$$\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r}}$$

radially symmetric harmonic functions:

We are looking for u harmonic and depending only on $r \Rightarrow u(x,y) = u(|(x,y)|) = u(r)$

Then we solve

$$r \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} u = 0 \right] \qquad \text{since } \frac{\partial}{\partial \theta} u = 0$$

$$\Updownarrow$$

$$(r u_r)_r = 0 \iff r u_r = c_0 \iff u_r = \frac{c_0}{r}$$

$$\iff u(r) = c_0 \log r + c_1$$

So in 2-D a radially symmetric harmonic function is of the form

$$u(x,y) = C_0 \log (x^2+y^2)^{\frac{1}{2}} + C_1$$

for $(x,y) \neq 0$!

Invariance in 3-D:

A similar analysis can be done in 3D

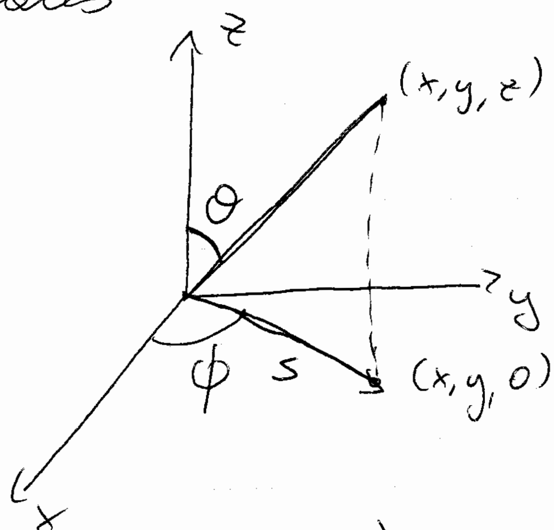
- Invariance under translation and rotation
- the Δ in spherical coordinates

$$x = s\cos\phi$$

$$y = s\sin\phi$$

$$z = r\cos\theta$$

$$s = r\sin\theta$$



$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + $$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

A radially symmetric harmonic function in $\mathbb{R}^3$ takes the form

$$r^2\left[u_{rr} + \frac{2}{r}u_r = 0\right]$$

$$(r^2u_r)_r = 2ru_r + r^2u_{rr} \Rightarrow u = -c_1r^{-1} + c_2$$

when $r = (x^2+y^2+z^2)^{\frac{1}{2}}$.