Lecture #11 10/18

Fourier transform and PDE

Diffusion Equation:

~~Recall that we already~~ Consider the problem

$$(D)\begin{cases} u_t = k u_{xx} \\ u\big|_{t=0} = \phi(x) \end{cases} \quad \left(\text{we assumed } \lim_{|x|\to\infty} \phi(x) = 0\right)$$

We proved that the unique solution for this problem is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy \quad \text{for } t>0$$

$$= S(t) * \phi(x)$$

here

$$S(x,t) = \frac{1}{(4\pi k t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4kt}} \qquad t>0$$

Recall that we found S by first solving

$$(D_1)\begin{cases} Q_t = k Q_{xx} \\ Q(0,x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \end{cases}$$

homogeneity $\Rightarrow$ ODE $\Rightarrow$ $Q(x,t)$, then

$$S(x,t) = \frac{\partial}{\partial x} Q(x,t)$$

then take derivative w.r.t. x everywhere in (D_1) (in the sense of distributions). Then

$$\begin{cases} (Q_x)_t = k\,(Q_x)_{xx} \\ Q_x(0,x) = \delta_0 \end{cases}$$

in fact $Q(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

then

$$(Q_x, \varphi) = -(Q, \varphi_x) = -\int Q(x)\,\varphi'(x)\,dx$$

$$= -\int_0^\infty \varphi'(x)\,dx = -\underset{\substack{\| \\ 0}}{\varphi(\infty)} + \varphi(0) = (\delta_0, \varphi)$$

So $Q_x = S(x,t)$ is the solution of the problem

$$(D_2)\quad \begin{cases} S_t = k\,S_{xx} \\ S(0,x) = \delta_0 \end{cases}$$

Let's find S using F.T.

Take (D_2) and take F.T. w.r.t the ~~x~~ variable.

Then

$$\mathcal{F}(S_t) = (\mathcal{F}(S))_t = (\hat{S})_t \quad \text{check this}$$

(*) Recall that $\sqrt{\frac{\pi}{a}}\, e^{-\frac{1}{4a}\xi^2} = \left(e^{-x^2 a}\right)^{\wedge}(\xi)$

hence if $a = (4kt)^{-1}$

$$e^{-kt\xi^2} = \left(\sqrt{\pi 4kt}\right)^{-1}\sqrt{\pi 4kt}\; e^{-kt\xi^2}$$

$$= \left(\frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{x^2}{4kt}}\right)^{\wedge}(\xi)$$

So that

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{x^2}{4kt}}$$ as we knew.

$$\begin{cases} \hat{S}_t = k\,\widehat{S_{xx}} \\ \hat{S}(0,\xi) = \hat{\delta}_0(\xi) \equiv 1 \end{cases}$$

Then recall that

$$\widehat{S_{xx}}(\xi) = (i\xi)^2\,\hat{S}(\xi) = -\xi^2\,\hat{S}(\xi)$$

$$\begin{cases} \hat{S}_t = -k\xi^2\,\hat{S} \\ \hat{S}(0,\xi) = 1 \end{cases}$$

For fixed ξ in $\mathbb{R}$ this is now an ODE with solution

$$\hat{S}(\xi,t) = 1\,e^{-k\xi^2 t} = e^{-kt\xi^2}$$

see back previous page 

Wave equation

Consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u|_{t=0} = 0 \quad u_t|_{t=0} = \phi(x) \end{cases}$$

then we proved that

$$u(x,t) = S(t) * \phi(x)$$

where

$$S(x,t) = \begin{cases} 0 & |x| > ct \\ \frac{1}{2c} & |x| \leq ct \end{cases}$$

then we found $S(x,t)$ by solving

$$\begin{cases} S_{tt} - c^2 S_{xx} = 0 \\ S(0,x) = 0 \quad S_t(0,x) = \delta_0(x) \end{cases}$$

let's now use the F.T

$$\widehat{S}_{tt} - c^2 \widehat{S_{xx}} = 0$$

$$\widehat{S}(0,\xi) = 0 \quad \widehat{S}_t(0,\xi) = 1$$

$$\widehat{S}_{tt} + c^2 \xi^2 \widehat{S} = 0$$

$\cancel{\widehat{S}(\xi,t)} =$

Consider the ODE

$$f_{tt} + \alpha f = 0$$

$$f(t) = \sin(t\beta)$$

$$f_{tt} = -\beta^2 \sin(t\beta)$$

$$-\beta^2 \sin(t\beta) + \alpha \sin(t\beta) = 0$$

$$\alpha = \beta^2 \qquad \beta = \sqrt{\alpha}$$

$$f(t) = A\sin(\sqrt{\alpha}\, t)$$

So above

$$\hat{S}(t,\xi) = A\sin(c\xi t)$$

$$\hat{S}(0,\xi) = A\sin(0) = 0$$

$$\hat{S}_t(t,\xi) = A\cos(c\xi t)\, c\xi$$

$$\hat{S}_t(0,\xi) = A\cos(0)\, c\xi = Ac\xi = 1$$

so $A = \frac{1}{c\xi}$

$$\hat{S}(t,\xi) = \frac{1}{c\xi}\sin(c\xi t) = \frac{1}{2ic\xi}\left[e^{ic\xi t} - e^{-ic\xi t}\right]$$

So now recall that $\widehat{\left(e^{i a \cdot \xi}\right)}$

$$e^{-a i \xi} \hat{f}(\xi) = \widehat{f(x-a)}$$

$$e^{a i \xi} \hat{f}(\xi) = \widehat{f(x+a)}$$

$$\frac{1}{i\xi} = \frac{1}{2}\widehat{\left(\operatorname{sign}(x)\right)}$$

$$\frac{1}{2c}\frac{1}{i\xi} e^{ict\xi} = \frac{1}{2c}\frac{1}{2}\widehat{\operatorname{sign}(x+ct)}$$

so

$$\hat{S}(t,\xi) = \frac{1}{4c}\left[\widehat{\operatorname{sign}(x+ct) - \operatorname{sign}(x-ct)}\right]$$

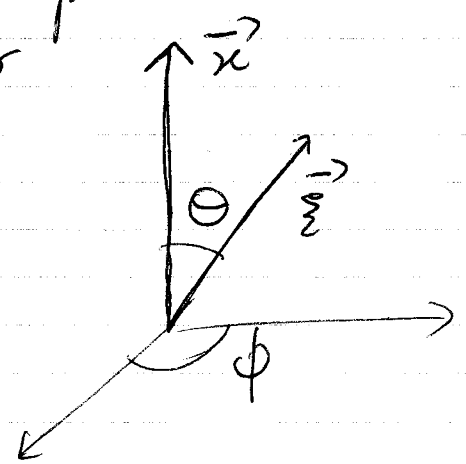
$$= \begin{cases} 0 & |x| > ct \\ \frac{1}{2c} & |x| < ct. \end{cases}$$

Now consider the 3D case. In a similar fashion

$$\begin{cases} \hat{S}_t = -c^2(\xi_1^2+\xi_2^2+\xi_3^2)\hat{S} \\ \hat{S}(\xi,0)=0 \quad \hat{S}_t(\xi,0)=1 \end{cases}$$

$$S(x,t) = \iiint \frac{1}{|\xi| c}\sin(|\xi| ct) e^{i\xi\cdot x}\frac{d\xi}{(2\pi)^3}$$

change in spherical coordinates



$|\xi| = K$

$0 < k < \infty$

$0 < \Theta < \pi$

$0 < \phi < 2\pi$

$$\xi \cdot x = |x| K \cos\Theta$$

$$= \frac{1}{(2\pi)^3}\frac{1}{c}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} K^{-1}\sin(kct)e^{ikr\cos\Theta} K^2\sin\Theta \, dk\, d\Theta\, d\phi$$

$$= \frac{1}{(2\pi)^2}\frac{1}{c}\int_0^\infty \sin(kct)\left(\int_0^\pi e^{ikr\cos\theta}\sin\theta\,d\theta\right)k\,dk$$

$$\int_0^\pi e^{ikr\cos\theta}\sin\theta\,d\theta =$$

$$-\frac{e^{ikr\cos\theta}}{ikr}\Big|_0^\pi = \frac{-e^{-ikr}+e^{ikr}}{ikr} = 2\frac{\sin(kr)}{kr}$$

$$-\cancel{\cos(-kr)} + i\sin(-kr) + \cancel{\cos(kr)} + i\sin(kr)$$

$$= \frac{1}{2\pi^2}\frac{1}{c}\int_0^\infty \sin(kct)\,\frac{\sin(kr)}{\cancel{k}r}\,\cancel{k}\,dk$$

$$= \frac{1}{2\pi^2 cr}\int_0^\infty \sin(kct)\sin(kr)\,dk$$

$$= \frac{1}{8\pi^2 cr}\mathrm{Re}\int_0^\infty \left[e^{ik(ct-r)} - e^{ik(ct+r)}\right]dk$$

$$= \frac{1}{4\pi cr}\left[\delta(ct-r) - \underbrace{\delta(ct+r)}_{=0}\right]$$

Ex in book

$\delta(ct+r) = 0$ because $ct+r>0$ for $t>0$

$$= \frac{1}{4\pi cr}\delta(ct-r) = \frac{1}{4\pi c^2 t}\delta(ct-r)$$

$\neq 0$ only when $r=ct$

Consequence:

~~Computation~~ Recall the following: if

$$u(x,t) = S(t) * \varphi(x)$$

$$u_{tt}(x,t) = S_{tt} * \varphi(x)$$

$$\Delta u(x,t) = \Delta S * \varphi(x)$$

so $u(x,t)$ solves the equation

$$\lim_{t\to 0} S(t) * \varphi(x) = \lim_{t\to 0} \int S(x-y,t)\,\varphi(y)\,dy$$

$$= \lim_{t\to 0} (S(\cdot,t), \varphi) = 0$$

$$u_t(x,t) = S_t * \varphi(x)$$

$$\lim_{t\to 0} S_t(x) = \delta_x$$

as a distribution

So $\lim_{t\to 0} u_t(x,t) = \phi(x)$

More importantly

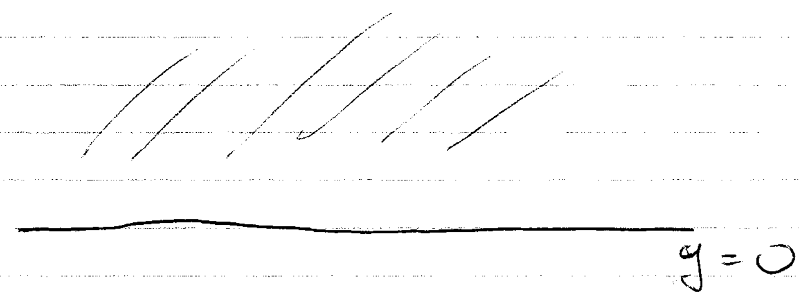
$$u(x,t) = \int \frac{1}{4\pi c^2 t}\,\delta(ct - |x-y|)\,\phi(y)\,dy$$

which says that the domain of dependence is on the surface of the cone!

Prove ~~Go back to~~ (*)

Laplace equation in Half-Plane

$$\begin{cases} u_{xx}+u_{yy}=0 & y>0 \\ u(x,0)=\delta(x) & y=0 \end{cases} \implies u \text{ is harmonic in half plane}$$



take F.T. w.r.t. x of equation after

$\cancel{\xi^2 \hat{u}}$ defining $U(\xi,y)=\mathcal{F}_x(u(x,y))(\xi)$

$$= \int_{-\infty}^{\infty} e^{-ix\xi}\, u(x,y)\, dx$$

~~the equation becomes~~ Also observe that

$\cancel{-\xi^2 U(\xi,y) + U}$

$$U_{yy}(\xi,y) = \partial_{yy}\int_{-\infty}^{\infty} e^{-ix\xi}\, u(x,y)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-ix\xi}\, u_{yy}(x,y)\, dx$$

$$= \mathcal{F}_x(u_{yy})(\xi)$$

then the equation becomes

$$\begin{cases} -\xi^2 U(\xi,y) + U_{yy}(\xi,y) = 0 \\ U(\xi,0) = 1 \end{cases}$$

Fix ξ and solve the ODE w.r.t. y

$$c e^{\alpha y} = U(y) \qquad U_{yy} = c\alpha^2 e^{\alpha y}$$

plug in

$$-\xi^2 e^{\alpha y} + \alpha^2 e^{\alpha y} = 0 \qquad \alpha^2 = \xi^2$$

$$\alpha = \pm|\xi|$$

$$U_1(y) = c e^{-|\xi| y} \qquad U_2 = c e^{|\xi| y} \qquad y > 0$$

$$U_1(0) = U_2(0) = 1 \implies c \equiv 1$$

but we want $U(y,\xi)$ to have inverse F.T. so we cannot pick $U_2(\xi,y) = e^{|\xi| y} \xrightarrow[|\xi|\to\infty]{} \infty$

so $$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi} e^{-|\xi| y}\, d\xi$$

$$= \frac{1}{2\pi} \int_0^\infty e^{ix\xi} e^{-\xi y} \, d\xi$$

$$+ \frac{1}{2\pi} \int_{-\infty}^0 e^{\xi(ix+y)} \, d\xi$$

$$= \frac{1}{2\pi} \left[\left. \frac{e^{\xi(ix-y)}}{(ix-y)} \right|_0^\infty + \left. \frac{e^{\xi(ix+y)}}{(ix+y)} \right|_{-\infty}^0 \right]$$

$$= \frac{1}{2\pi} \left[-\frac{1}{ix-y} + \frac{1}{ix+y} \right] = \frac{1}{2\pi} \left[\frac{2y}{x^2+y^2} \right]$$

So the source function in this case is

$$S(x,y) = \frac{y}{x^2+y^2}$$

If now we consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & y > 0 \\ u(x,0) = \phi(x) \end{cases}$$

then

$$u(x,y) = \int_{-\infty}^{\infty} S(x-\xi, y) \, \phi(\xi) \, d\xi \quad !$$