Lecture #6

Final Exam :

Thursday December 16
9:00 – 12:00 AM

We recall from last time that The solution of the initial Value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u|_{t=0} = \phi(x) \\ \dot{u}|_{t=0} = \psi(x) \end{cases}$$

Can be written as
$$\boxed{u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}$$

Out of curiosity in higher dimensions we have in $\mathbb{R}^2$

$$u(x,y,t) = \int_D \frac{\psi(\xi, w)}{[c^2t^2 - (x-\xi)^2 - (y-w)^2]^{\frac{1}{2}}} \frac{d\xi\, dw}{2\pi c}$$

$$+ \frac{\partial}{\partial t} \int_D \frac{\phi(\xi, w)}{[c^2t - (x-\xi)^2 - (y-w)^2]^{\frac{1}{2}}} \frac{d\xi\, dw}{2\pi c}$$

$$D = \{ (x-\xi)^2 + (y-w)^2 \leq c^2t^2 \}$$

in $\mathbb{R}^3$

$$u(x,y,z,t) = \frac{1}{4\pi^2 c^2 t}\int_S \psi(\xi,w,\tau)\,d\sigma$$

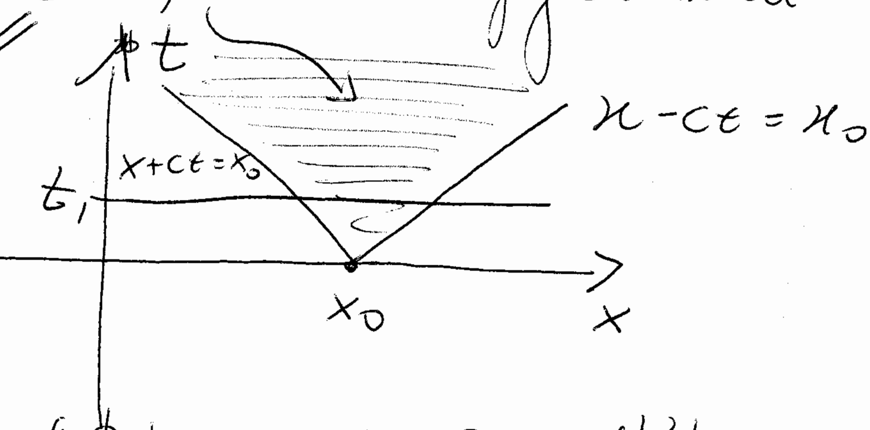
$$+\frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_S \phi(\xi,w,\tau)\,ds\right]$$

where $S = \{\, |(x,y,z)-(\xi,w,\tau)| = ct \,\}$

Causality and Energy

Principle of causality: An initial condition (ϕ,ψ) at a point $(x_0,0)$ can effect the solution $u(x,t)$, $t>0$ only in the set $\delta(x_0,t)$ in the figure below

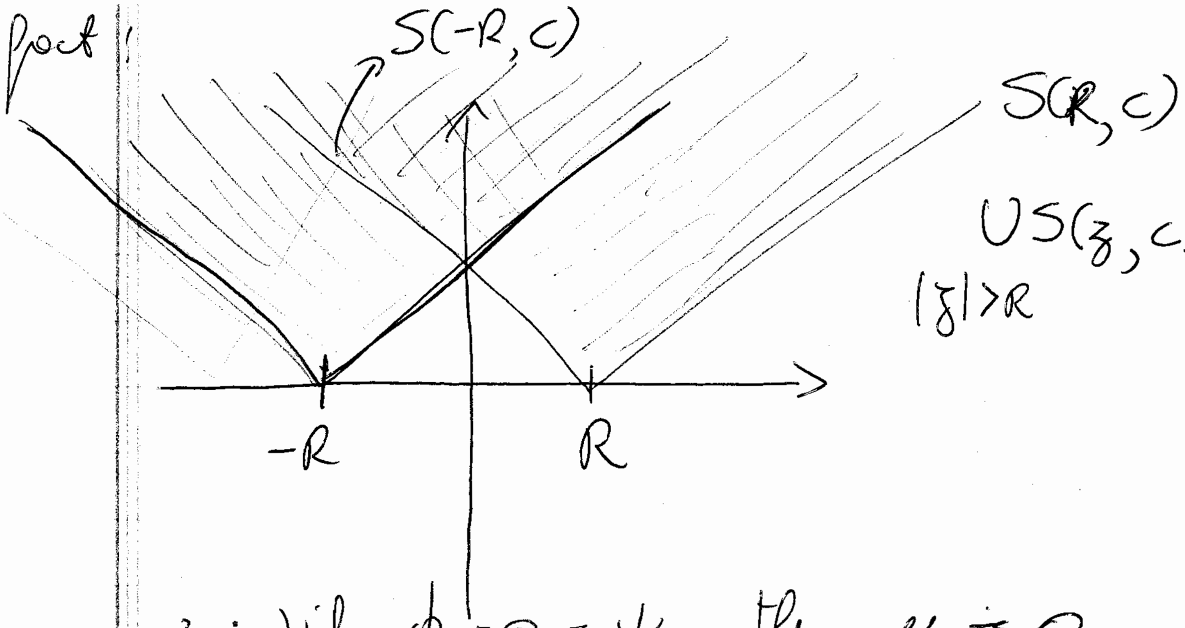
$\{\, |x-x_0| \le ct,\ t>0 \}$



Consequences: if $\phi\big|_{|x|>R} = 0$ $\psi\big|_{|x|>R} = 0$

then the solution $u(x,t) = 0$ on $|x| > R+ct$

In fact:



$\bigcup_{|\xi|>R} S(\xi, c) = \{ |x| > R+c$

Consequence 2: a) if $\phi \equiv 0 \equiv \psi$ then $u \equiv 0$ (uniqueness)

b) if $\phi_1 = \phi_2$, $\psi_1 = \psi_2$ in

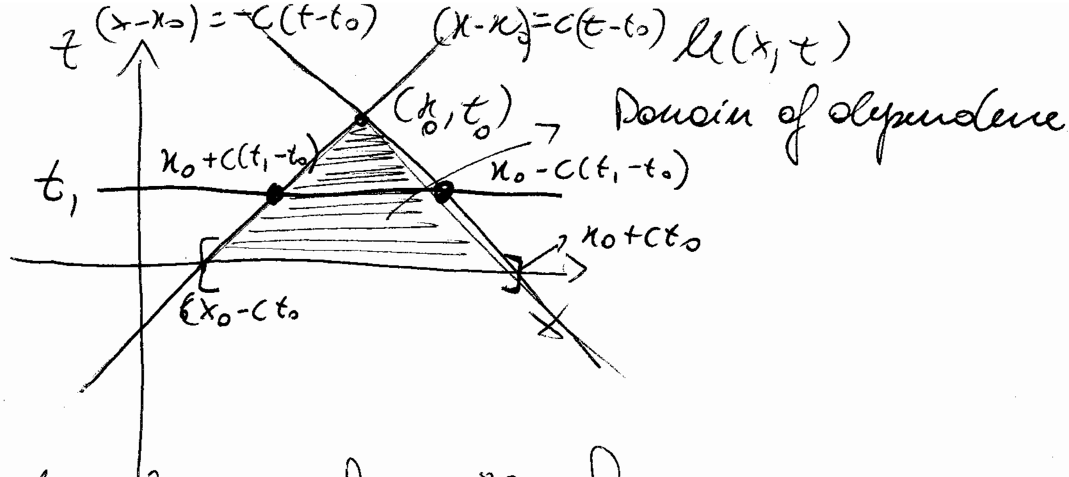
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u\big|_{t=0} = \phi_1 \quad \dot{u}\big|_{t=0} = \psi_1 \end{cases} \quad \& \quad \begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v\big|_{t=0} = \phi_2 \, , \dot{u}\big|_{t=0} = \psi_2 \end{cases}$$

then $u \equiv v$

In fact $u - v = w$ solves $\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w\big|_{t=0} = \dot{w}\big|_{t=0} = 0 \end{cases}$

and from a) $\Rightarrow w \equiv 0$

Definition: the $\boxed{\text{domain of dependence}}$ or past history of the point (x_0, t_0) is the region in (x,t) on which the value $u(x_0, t_0)$ depends upon;



In particular ~~on~~ from the formula

$$u(x,t)=\frac{1}{2}\Big(\phi(x-ct)+\phi(x+ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)\,d\xi$$

one can see that the value of u at (x_0,t_0) is influenced only by the value of ϕ at (x_0-ct_0) and (x_0+ct_0) and the value of ψ on $[x_0-ct_0, x_0+ct_0]$

In analogy if $t_1<t_0$ the value of $u(x_0,t_0)$ is only influenced by $u(x_0+c(t_1-t_0))$ and $u(x-c(t_1-t_0))$ and $\mathring{u}$ on $[x_0+c(t_1-t_0), x_0-c(t_1-t_0)]$.

④ A proof of the principle of causality for $\phi=0$ $\psi=\delta_{x_0}$

We already proved that in this case the solution is

$$S_{x_0}(x,t)=\begin{cases}0 & \text{if } |x-x_0|>ct\\ \frac{1}{2c} & \text{if } |x-x_0|\le ct\end{cases}$$

The conservation of energy

Consider an "infinite string" of constant density ρ and constant tension vector T

$\rho u_{tt} - T u_{xx} = 0$

The kinetic energy of the string is

$$k_e = \frac{1}{2}\rho\int_{-\infty}^{\infty} u_t^2\,dx \quad \left(\text{coming from } \frac{1}{2}mv^2\right)$$

The potential energy

$$P_e = \frac{1}{2}T\int_{-\infty}^{\infty} u_x^2\,dx \qquad \Rightarrow \quad E(t) = k_e + P_e\,(t)$$

Assume that the initial data ϕ and ψ are zero outside $[-R, R]$. From Consequence 1 above $\Rightarrow$

$$u,\ \dot{u} \equiv 0 \quad \text{for } |x| > R + ct$$

We want to show that

$$E(t) = E(0)$$

To prove this it's enough to show that

$$\frac{d}{dt}E(t) = 0$$

$$\frac{d}{dt}E = \frac{d}{dt}k_e + \frac{d}{dt}P_e$$

(6)

$$\frac{d}{dt} K_e = \frac{\rho}{2} \frac{d}{dt} \int_{-\infty}^{\infty} u_t^2 \, dx$$

we can pass the derivative in thanks to smoothness of u and compact support of $u(x,t)$ in $[-R-ct, R+ct]$

$$\| $$

$$\frac{1}{\cancel{2}} \rho \cancel{2} \int_{-\infty}^{\infty} u_t u_{tt} \, dx$$

now $u_{tt} = \frac{T}{\rho} u_{xx}$

$$= \cancel{\rho} \int_{-\infty}^{\infty} u_t \frac{T}{\cancel{\rho}} u_{xx} =$$

$u_t u_{xx} = \partial_x (u_t u_x) - (u_{tx} u_x)$

~~integration by parts~~

$$= T \int_{-\infty}^{\infty} \partial_x (u_t u_x) \, dx - T \int_{-\infty}^{\infty} u_{tx} u_x \, dx$$

$\|$

0 because perfect derivative

and

$$\lim_{x \to \pm\infty} u_t u_x = 0$$

On the other hand

$$\frac{d}{dt} P_e = \frac{1}{\cancel{2}} T \cancel{2} \int_{-\infty}^{\infty} u_x u_{xt}$$

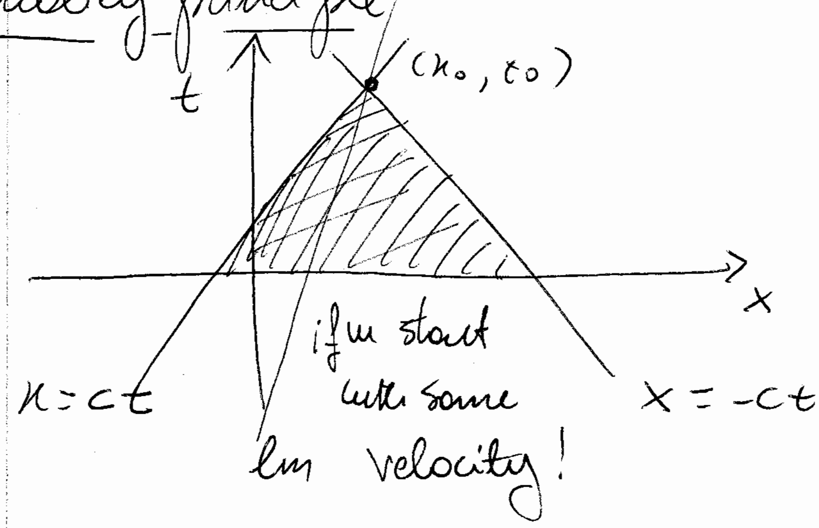
So
$$\frac{d}{dt} E = -T \int_{-\infty}^{\infty} u_{tx} u_x \, dx + T \int_{-\infty}^{\infty} u_x u_{xt} = 0$$

Q.E.D.

The fact that $E(t) = E(0)$ is called <u>conservation</u> of <u>energy</u>. This is one of the most basic facts about the wave equation.

Few words on $\boxed{\text{Huygen's principle in 3D}}$

1-D Causality principle



Let's line be any line through (x_0, t_0) in the domain of influence. Then

$$|m| = |\text{slope of line}| \leq c$$

~~If we think of c = speed of light then the components of the electric and magnetic field are solutions to wave eqs with c = speed of light. If we go back~~

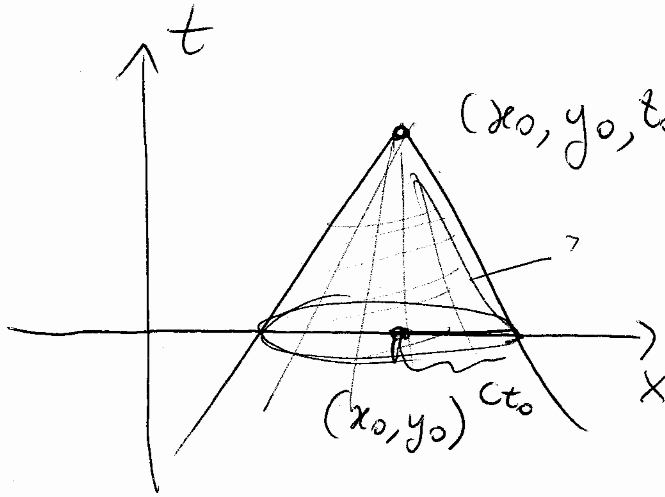
So a signal traveling from (x_0, t_0) subject to the equation $u_{tt} - c^2 u_{xx} = 0$ cannot move at

a speed higher than c, but it could have any
other speed $\le c$.

A similar situation is true in 2D if one looks at the formula for the solution:

$$u(x,y,t) = \int_D \frac{\psi(\xi,w)}{[c^2t^2-(x-\xi)^2-(y-w)^2]^{\frac{1}{2}}} \frac{d\xi\, dw}{2\pi c} + \frac{\partial}{\partial t}\int_D \frac{\phi(\xi,w)}{[\quad]^{\frac{1}{2}}} \frac{d\xi\, dw}{2\pi c}$$

$$D = \{ (x-\xi)^2 + (y-w)^2 \le c^2t^2 \}$$



The domain of influence is the whole cone.

So in particular if one lived on a 2D (flat) space life as we know it, would be very unpleasant. We would ~~hear our own voi~~ hear parts of sentences spoken at a early time with different velocities. All the images would be mixed up.

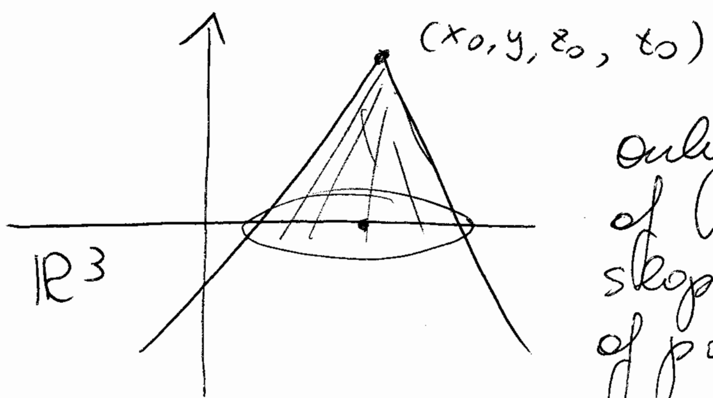
In 3D instead

$$u(x,y,z,t) = \frac{1}{2\pi^2 c^2 t}\int_S \psi(\xi,w,\tau)\,d\sigma$$

$$+\frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_S \phi(\xi,w,\tau)\,d\sigma\right]$$

$$S = \left\{ |(x,y,z) - (\xi,w,\tau)| = ct \right\}$$

this means that ~~no matt~~ the Domain of Influence is



only the boundary of the cone, so the slope of the trajectories of possible signs is always $= c$.

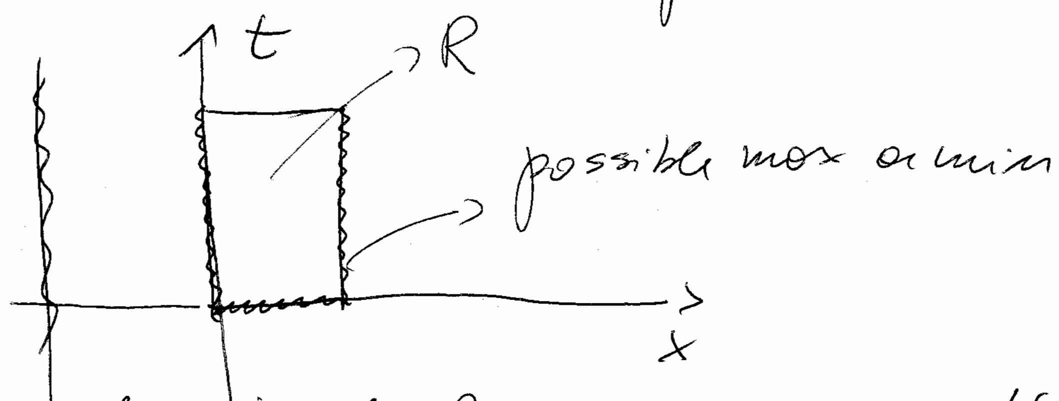
In other words the speed ~~is always eq~~ of the signal, no matter what is the initial data (ϕ, ψ) is $= c$. ~~the~~ So different signals ~~traveling~~ "vision" ~~obeying~~ do not get overlapped $\Rightarrow$ we see clearly the way we do so for sounds!

~~Proof~~ The ~~causality~~ causality principle in 3D is called ~~Huy~~ $\boxed{\text{Huygens's principle}}$.

The diffusion equation:

We consider (*) $u_t = k u_{xx}$ $\quad$ k = constant

Maximum Principle : if u solves (*) in a rectangle ($0 \le x \le l$, $0 \le t \le T$) in space-time, then the maximum value of $u(x,t)$ is assumed either at $t=0$ or at $x=0$ or at $x=l$. Same for minimum value



Assume max u when inside $R = [0,l] \times [0,T]$, at (x_0, t_0)

$$\text{So} \quad u_x(x_0,t_0) = 0 \qquad u_t(x_0,t_0) = 0$$
$$u_{xx}(x_0,t_0) \le 0$$

If we knew that ~~$u_{xx} \neq 0$~~ $u_{xx}(x_0,t_0) \neq 0$ then

$u_{xx}(x_0,t_0) < 0$ which together with
will give $k u_{xx} < 0 = u_t$ at $(x_0,t_0) \Rightarrow$ cont.

But we do not know that $u_{xx}(x_0, t_0) \neq 0$!

Now consider $u(x,t) + \varepsilon x^2 = v(x,t)$

(~~Notice that x^2 is known~~ ε is a very small constant

Also assume that $M = \max\limits_{\partial_u R} u$

$$\partial_u R = \begin{cases} R \cap \{x=0\}; \\ R \cap \{x=\ell\} \\ R \cap \{t=0\} \end{cases}$$

We want to prove that $u(x,t) \le M \quad \forall (x,t) \text{ in } R$!

Suppose we could prove that

$$(\star\star) \quad v(x,t) \le M + \varepsilon \ell^2 \quad \text{for any } (x,t) \text{ in } R$$

Then $\qquad u(x,t) \le M + \varepsilon(\ell^2 - x^2) \le M + \varepsilon \ell^2 \xrightarrow{\varepsilon \to 0} M$ Q.E.D.

because this is true for any $\varepsilon > 0$

We are left with proving $(\star\star)$

$$v\big|_{x=\ell} = u\big|_{x=\ell} + \varepsilon \ell^2 \le M + \varepsilon \ell^2$$

$$v\big|_{x=0} = u\big|_{x=0} \le M \qquad \Longrightarrow \qquad v\big|_{\partial R} \le M + \varepsilon \ell^2$$

$$v\big|_{t=0} = u\big|_{t=0} + \varepsilon x^2 \le M + \varepsilon \ell^2$$

Now let's prove that

$$(**)\quad \max_{(x,t)\text{ in } \mathring{R}} v(x,t) \le \max_{\partial R} v \left(\le M+\varepsilon \ell^2\right)$$

If the max where in $\mathring{\ell}$, say at (x_0,t_0) then $v_t\big|_{(x_0,t_0)} = v_x\big|_{(x_0,t_0)} = 0 \quad v_{xx}\big|_{(x_0,t_0)} \le 0$

$$\underbrace{v_t - k v_{xx}}_{\ge 0} = u_t - k u_{xx} - 2\varepsilon k = -2\varepsilon k < 0$$

$\Longleftrightarrow$ contradiction

~~Suppose now that~~ So we know that

$$\max_R v = \max_{\partial R} v$$

If $\max_{\partial R} v = \max_{\partial_L R} v \Rightarrow$ done

so suppose

$$\max_{\partial R} v = v(x_1, T) \qquad 0 < x_1 < \ell$$

$v_x(x_1, T) = 0 \qquad v_{xx}(x_0, T) \le 0$. What about $v_t(x_0,T)$

$$v_t(x_0,T) = \lim_{\delta \to 0^+} \frac{v(x_0,T) - v(x_0,T-\delta)}{\delta} \ge 0$$

but by definition of max $v(x_0,T) \ge v(x_0,T-\delta)$

so at (x_0,T) $\underbrace{v_t - k v_{xx}}_{\ge} = -2\varepsilon k < 0 \Rightarrow \#$ Q.E.D.