18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 7

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 68.1 $u_t - ku_{xx} = f(x,t), x,t > 0$
$u(0,t) = 0$
$u(x,0) = \varphi(x)$
Extend φ and f to be odd functions of x:

$$\varphi_{\text{odd}}(x) = \begin{cases} \varphi(x), & x > 0 \\ -\varphi(-x) & x < 0, \end{cases} \qquad f_{\text{odd}}(x) = \begin{cases} f(x,t), & x > 0 \\ -f(-x,t) & x < 0, \end{cases}$$

Let ν be the solution of

$$\begin{aligned} \nu_t - k\nu_{xx} &= f_{\text{odd}}(x,t) \quad (x,t) \in \mathbb{R} \times (0,\infty) \\ \nu(x,0) &= \varphi_{\text{odd}}(x) \end{aligned}$$

Then ν is an odd function of x. (Indeed, $w(x,t) = \nu(x,t) + \nu(-x,t)$ solves $w_t - kw_{xx} = 0$, $w(x,0) = 0$, now use uniqueness.) Thus $\nu(0,t) = 0$ for $t > 0$, so $u(x,t) = \nu(x,t), x > 0$ solves our Dirichlet problem. Explicitly,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{odd}}(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds.$$

Divide up each integral with respect to y into two parts $\int_0^\infty (\quad)\,dy + \int_{-\infty}^0 (\quad)\,dy$. Change variables $y \to -y$ in the second integral, and use $\varphi_{\text{odd}}(y) = -\varphi_{\text{odd}}(-y), f_{\text{odd}}(y,s) = -f_{\text{odd}}(-y,s)$ there. Thus, we get

$$u(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\varphi(y)dy + \int_0^t \int_0^\infty [S(x-y,t-s) - S(x+y,t-s)]f(y,s)dyds.$$

Problem 68.2

$$\begin{cases} \nu_t - kv_{xx} &= f(x,t) \\ \nu(0,t) &= h(t) \\ \nu(x,0) &= \varphi(x) \end{cases}$$

Let $V(x,t) = \nu(x,t) - h(t)$, so

$$\begin{cases} V_t - kV_{xx} &= f(x,t) - h'(t) \\ V(0,t) &= 0 \\ V(v,0) &= \varphi(x) - h(0). \end{cases}$$

By problem 68.1,

$$\begin{aligned} V(x,t) &= \int_0^\infty [S(x-y,t) - S(x+y,t)](\varphi(y) - h(0))dy \\ &\quad + \int_0^t \int_0^\infty [S(x-y,t-s) - S(x+y,t-s)] \left(f(y,s) - h'(s)\right) dyds. \end{aligned}$$

$\nu(x,t) = V(x,t) + h(t)$. Some simplification can be done.

$$\int_0^\infty [S(x-y,t) - S(x+y,t)]h(0)dy = h(0)\int_0^\infty [S(x-y,t) - S(x+y,t)]dy = h(0)\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

by example 1 on p. 57.

Similar simplifcation can be performed on the $h'(s)$ part of the 2nd integeral, but that still doesn't make the result too transparent.

Problem 76.1

$$\begin{cases} u_{tt} &= c^2 u_{xx} + xt & \Rightarrow f(x,t) = xt \\ u(x,0) &= 0 \\ u_t(x,0) &= 0. \end{cases}$$

By theorem 1 on p. 69,

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} ysdy\right) ds \\ &= \frac{1}{2c}\int_0^t \frac{y^2 s}{2}\Big|_{y=x-c(t-s)}^{x+c(t-s)} ds \\ &= \frac{1}{2c}\int_0^t [(x+c(t-s))^2 - (x-c(t-s))^2]\cdot\frac{s}{2}ds \\ &= \int_0^t x(t-s)sds = x\left(\frac{ts^2}{2} - \frac{s^3}{3}\right)\Big|_{s=0}^t \\ &= \frac{xt^3}{6}. \end{aligned}$$

Problem 76.5 $u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)dyds$.
Let $w(x,t,s) = \int_{x-ct+cs}^{x+ct-cs} f(y,s)dy$.
Thus, $u(x,t) = \frac{1}{2c}\int_0^t w(x,t,s)ds$.
In particular, $u(x,0) = \frac{1}{2c}\int_0^0 w(x,0,s)ds = 0$.
Also, $u_t(x,t) = \frac{1}{2c}\int_0^t w_t(x,t,s)ds + \frac{1}{2c}w(x,t,t)$ and $w(x,t,t) = \int_x^x f(y,t)dy = 0$, so

$$u_t(x,0) = \frac{1}{2c}\int_0^0 w_t(x,0,s)ds = 0,$$

so u satisfies the initial conditions. Moreover,

$$u_{tt}(x,t) = \frac{1}{2c}\int_0^t w_{tt}(x,t,s)ds + \frac{1}{2c}w_t(x,t,t).$$

Now,

$$w_t(x,t,s) = cf(x+ct-cs,s) - (-c)f(x-ct+cs,s) = c(f(x+ct-cs,s)) + f(x-ct+cs,s).$$

So

$$w_{tt}(x,t,s) = c^2(f'(x+ct-cs,s)) - f'(x-ct+cs,s))$$

and

$$w_t(x,t,t) = c(f(x,t) + f(x,t)),$$

so

$$u_{tt}(x,t) = \frac{1}{2c}\int_0^t c^2(f'(x+ct-cs,s) - f'(x-ct+cs,s))ds + f(x,t).$$

On the other hand,

$$\begin{aligned} u_x(x,t) &= \frac{1}{2c}\int_0^t w_x(x,t,s)ds \\ &= \frac{1}{2c}\int_0^t (f(x+ct-cs,s) - f(x-ct+cs,s))ds \\ u_{xx} &= \frac{1}{2c}\int_0^t (f'(x+ct-cs,s) - f'(x-ct+cs,s))ds. \end{aligned}$$

Combining these shows that

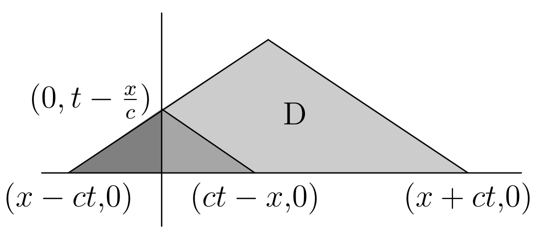
$$u_{tt} = c^2 u_{xx} + f(x,t)$$

indeed.

Problem 76.10 Given $f(x,t)$ for $x > 0, t > 0$, let $f_{\text{odd}}(x,t)$ be the odd extension of f in x (as in problem 68.1). Let ν be the solution of the wave equation $\nu_{tt} = c^2\nu_{xx} + f_{\text{odd}}$, with 0 initial data. Then ν is odd (as in 68.1), so $\nu(0,t) = 0$, so $u(x,t) = \nu(x,t), x, t > 0$ solves the Dirichlet problem. Explicitly,

$$u(x,t) = \frac{1}{2c}\int\int_{D'} f_{\text{odd}}$$

where D' is the domain of dependence for the whole line, i.e. the triangle with vertices $(x,t), (x+ct,0)$ and $(x-ct,0)$.



But f_{odd} is odd in x so the integrals over the two small triangles with vertices $(0, t-\frac{x}{c}), (0,0)$, and either $(x-ct,0)$ or $(ct-x,0)$ (shaded in the picture) are the negatives of each other, so they cancel. Hence

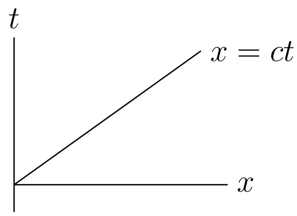
$$u(x,t) = \frac{1}{2c}\int\int_{D'} f_{\text{odd}} = \frac{1}{2c}\int\int_D f.$$

When D is the domain of dependence for the half line. Explicitly,

$$\int_{x-ct+cs}^{x+ct-cs} f_{\text{odd}}(y,s)dy = \int_{x-ct+cs}^{ct-cs-x} f_{\text{odd}}(y,s)dy + \int_{ct-cs-x}^{x+ct-cs} f_{\text{odd}}(y,s)dy.$$

The first integral is on an interval which is symmetric around 0; since f_{odd} is odd, the integral vanishes. The second term gives rise to the integral over D if integrated with respec to s as well.

Problem 76.14 One can do this directly, using even extensions. But it is easier to proceed as in the Dirichlet problem. Just as there (see p. 76), $u(x,t) = 0$ for $0 < ct < x$ (this is really just



uniqueness together with the fact that the solution depends only on initial data in the domain of dependence).
Thus $u(x,t) = j(x+ct) + g(x-ct)$ and $\varphi, \psi \equiv 0$ give $j(s) = g(s) = 0$ for $s > 0$. For $0 < x < ct$ we thus have $x + ct > 0$, so $j(x+ct) = 0$, so $u(x,t) = g(x-ct)$ there. Thus $u_x(x,t) = g'(x-ct)$, so $u_x(0,t) = g'(-ct)$. Since $u_x(0,t) = k(t)$, we deduce that

$$\begin{aligned} g'(-ct) &= k(t), t > 0; \\ g'(s) &= k\left(-\frac{s}{c}\right), s < 0. \end{aligned}$$

Thus

$$g(s) = \int_0^s g'(\sigma)d\sigma = \int_0^s k(-\frac{\sigma}{c})d\sigma.$$

Letting

$$g(s) = -c\int_0^{-s/c} k(\rho)d\rho,$$

we have

$$u(x,t) = g(x-ct) = -c\int_0^{t-\frac{x}{c}} k(\rho)d\rho, 0 < x < ct.$$